Inhomogeneous Universe Models & Backreaction

@UNESCO, Paris 2 July 2009

Lagrangian perturbation & Averaging of a general relativistic inhomogeneous universe

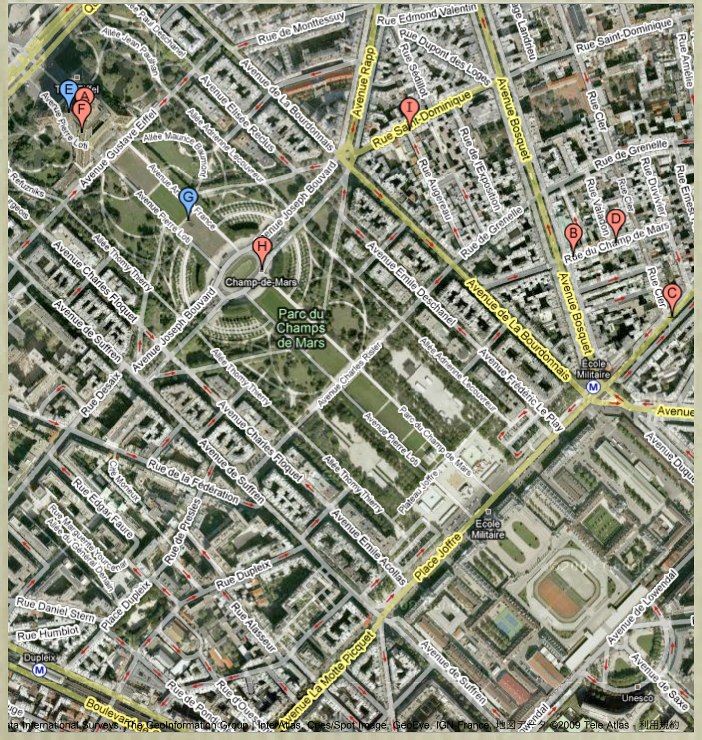
Hideki Asada
(Hirosaki U)

§1. Introduction

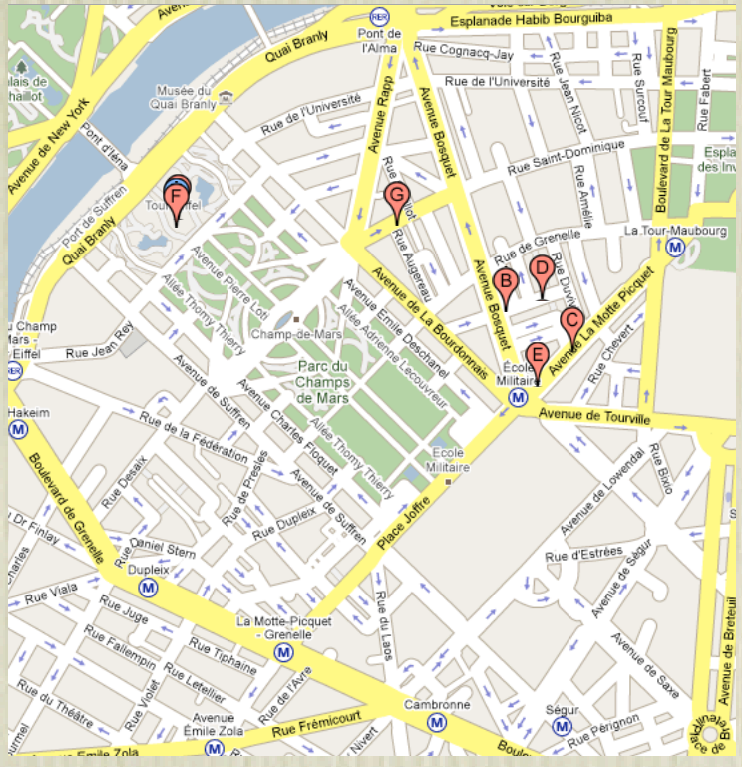
?

$\neq$

Evolution



Averaging



In General Relativity,

Einstein Eq

non-linear

"Averaging" is NOT trivial

(Long-standing) Problem

Averaging of

"**evolved**" universe

Evolution of

"**averaged**" universe

1) Both agree or not ?

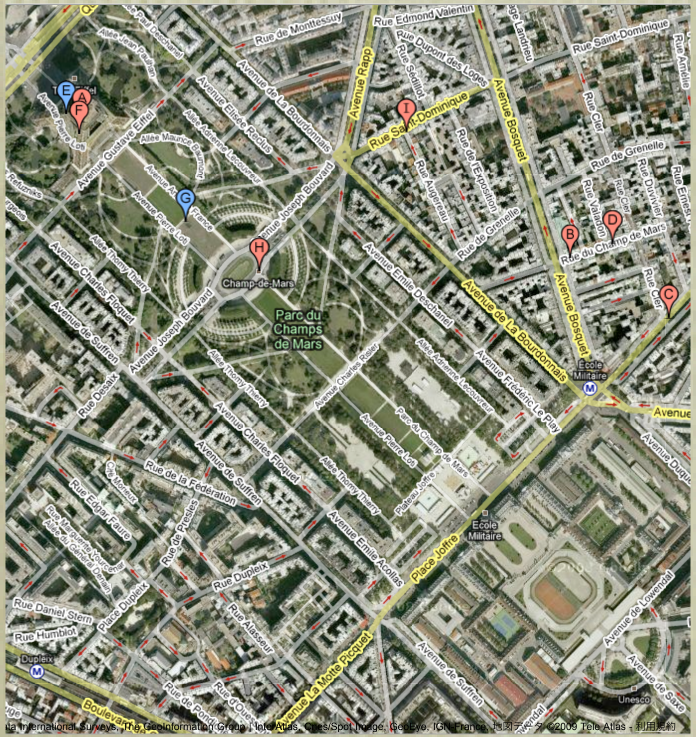
2) If not,

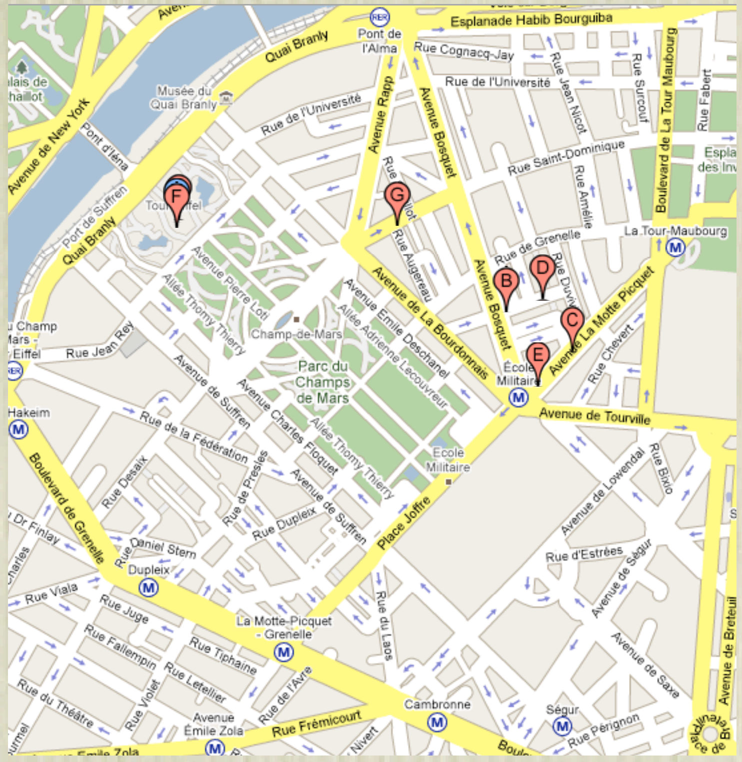
what's difference?

?

$\neq$

Evolution





Averaging

§ 1 Lagrange approach in GR

Euler vs Lagrange

Fluctuation Amplitude

$$\frac{\delta\rho}{\rho} \ll 1 \qquad \frac{\delta\rho}{\rho} << \infty$$

Newton vs GR

Fluctuation Wavelength

$$\frac{\ell}{L_H} \ll 1 \qquad All\ \ell$$

Therefore,

Best combination ...

Lagrange + **GR**

$$\frac{\delta\rho}{\rho} << \infty \quad \& \quad All\ \ell$$

It is nothing but

GR extension of Zeldovich approx.

A key is ...

$$\Delta\phi = 4\pi G(\rho - \rho_b)$$

$$\left(\frac{L}{l}\right)^2 \phi \sim \delta$$

Even when $\phi\ (g_{\mu\nu}\ in\ GR) < 1$

Large $\dfrac{L}{l}$ $\leftrightarrow$ **Large** δ

Dust-dominated universe

$$T^{\mu\nu} = \rho u^\mu u^\nu$$

$$T^{\mu\nu}{}_{;\nu} = 0$$

$$\Downarrow$$

$$\rho_{;\mu} u^\mu + \rho u^\mu{}_{;\mu} = 0$$

P.D.E.

$$u^\mu{}_{;\nu} u^\nu = 0$$

relativistic Beltrami equation

$$\left(\frac{\omega^\mu}{\rho}\right)_{;\nu} u^\nu = u^\mu_{;\nu}\left(\frac{\omega^\nu}{\rho}\right)$$

P.D.E.

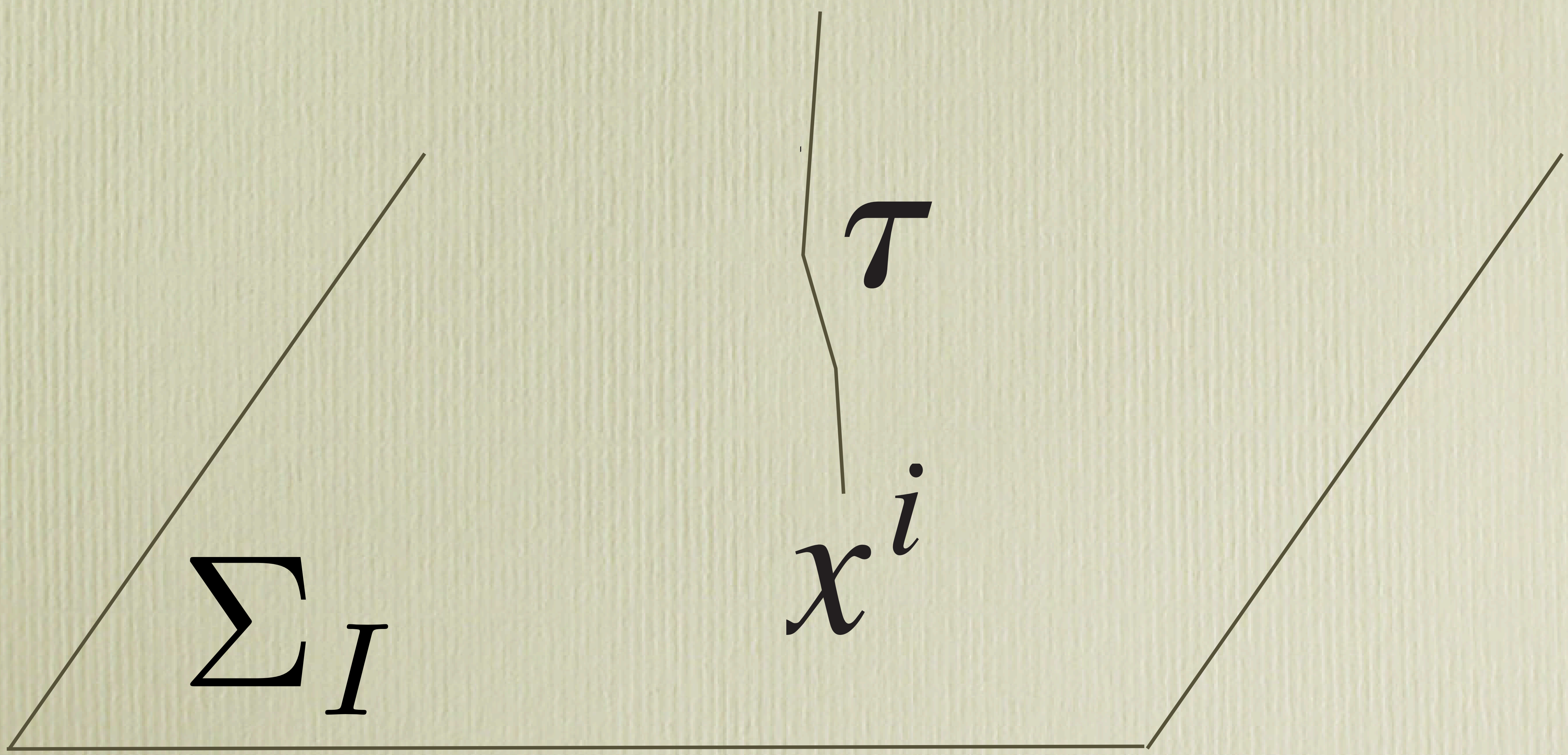
$\Updownarrow$ e.g., Ehlers (1961)

Beltrami equation **(Newton)**

$$\frac{d}{dt}\left(\frac{\omega}{\rho}\right) = \left(\frac{\omega\cdot\nabla}{\rho}\right)\mathbf{v}$$

Lagrange coordinate

$$x^{\mu} = (\tau, x^i)$$



$\Updownarrow$

Lagrange condition

$$u^{\mu} = (1,0,0,0)$$

If irrotational,
Comoving Synchronous

$$(\rho\sqrt{-g})_{,0}=0$$ **O.D.E !**

Exactly solved (formally)

$$\downarrow$$

$$\rho(\boldsymbol{x},t)=\sqrt{\frac{g(\boldsymbol{x},t_0)}{g(\boldsymbol{x},t)}}\rho(\boldsymbol{x},t_0)$$

HA, Kasai, PRD59, 123515(1999)

O.D.E !

$$\left(\frac{\omega^\mu}{\rho}\right)_{,0} = 0 \quad \Rightarrow \quad \frac{\omega^i}{\rho} = \left.\frac{\omega^i}{\rho}\right|_{t_0}$$

Kelvin-Helmholz Th. in curved spacetime

$$\omega^i(\boldsymbol{x},t) = \sqrt{\frac{g(\boldsymbol{x},t_0)}{g(\boldsymbol{x},t)}}\,\omega^i(\boldsymbol{x},t_0)$$

$$u^{\mu}{}_{;\nu} u^{\nu} = 0,$$ **P.D.E.**

$$\Downarrow$$

$$u_i = u_i(\boldsymbol{x}), \quad \text{and} \quad g_{0i} = g_{0i}(\boldsymbol{x})$$

dependent on spatial coordinates only

Einstein Eq.

Perturbatively solved

$$g_{\mu\nu} = g_{\mu\nu}^{FLRW} + h_{\mu\nu}$$

Assume $h_{\mu\nu} \ll 1$

Matter with pressure

$$T^{\mu\nu} = (\varepsilon + P) u^\mu u^\nu + P g^{\mu\nu}$$

$$T^{\mu\nu}{}_{;\nu} = 0$$

$$\Downarrow$$

P.D.E.

$$(\varepsilon u^\mu)_{;\mu} + P u^\mu{}_{;\mu} = 0$$

$$(\varepsilon + P) u_{\mu;\nu} u^\nu + P_{,\nu} \gamma^\nu_\mu = 0$$

non-geodesic motion

Therefore, one may ask

“can we choose Lagrange condition for such a non-geodesic?”

Yes, we can.

Barotropic EOS

$$P = P(\rho)$$

e.g., Ehlers (1961)

Entropy

$$s = \exp\left(\int \frac{d\varepsilon}{\varepsilon + P} \right)$$

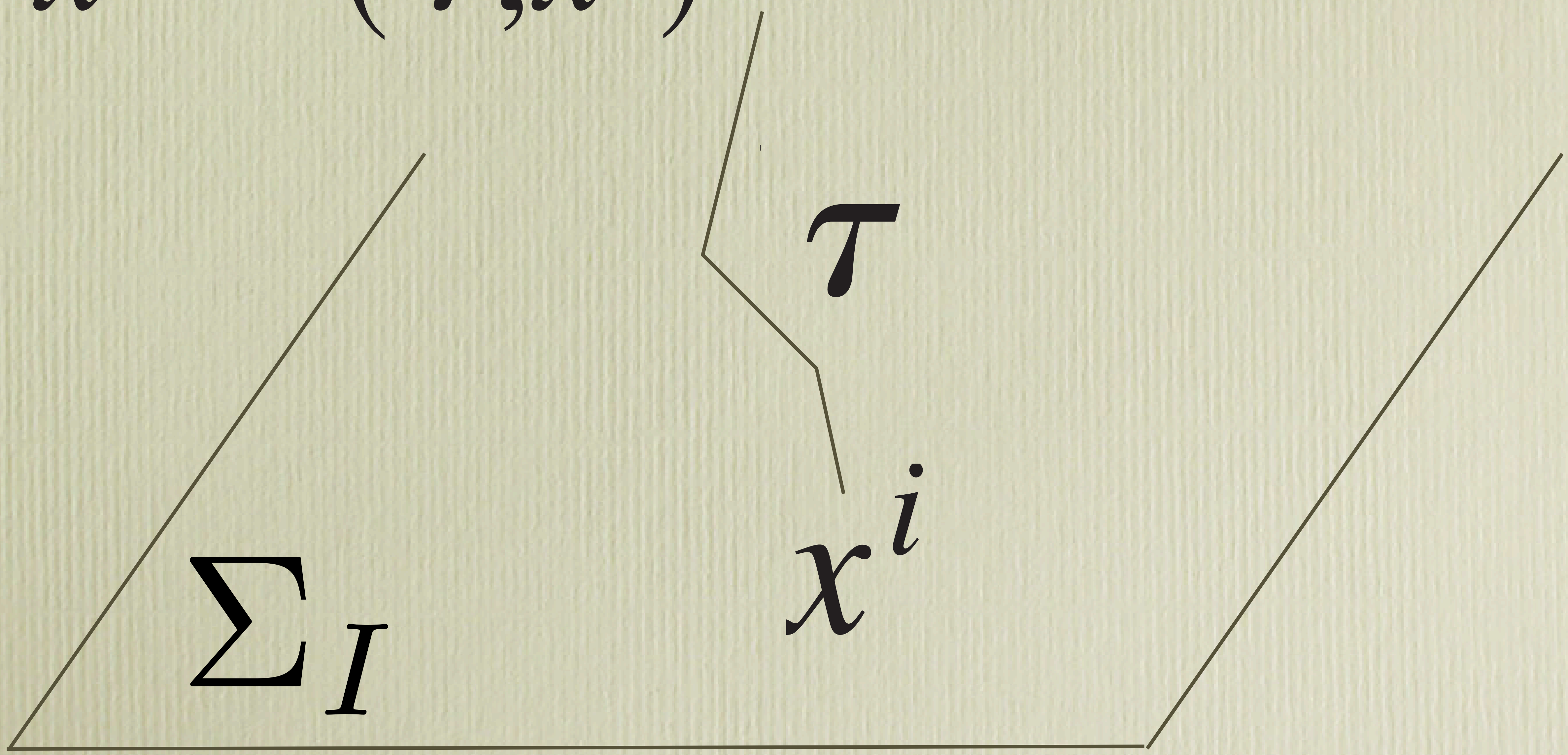
Enthalpy

$$h = \exp\left(\int \frac{dP}{\varepsilon + P} \right)$$

One can still choose

Lagrange coordinate

$$x^{\mu} = (\tau, x^{i})$$



relativistic Beltrami equation

$$\left(\frac{h\,\omega^{\mu}}{s}\right)_{;\nu}\frac{u^{\nu}}{h}=\left(\frac{u^{\mu}}{h}\right)_{;\nu}\frac{h\,\omega^{\nu}}{s}$$

P.D.E.

$\Updownarrow$ e.g., Ehlers (1961)

Beltrami equation **(Newton)**

$$\frac{d}{dt}\left(\frac{\omega}{\rho}\right)=\left(\frac{\omega\cdot\nabla}{\rho}\right)\mathbf{v}$$

Exactly solved (formally)

$$(\rho\sqrt{-g})_{,0}=0$$

$$(s\sqrt{-g})_{,0}=0$$

O.D.E !

$$\rho(\boldsymbol{x},t)=\sqrt{\frac{g(\boldsymbol{x},t_0)}{g(\boldsymbol{x},t)}}\rho(\boldsymbol{x},t_0)$$

$$s(\boldsymbol{x},t)=\sqrt{\frac{g(\boldsymbol{x},t_0)}{g(\boldsymbol{x},t)}}s(\boldsymbol{x},t_0)$$

HA, PRD62, 127301(2000)

O.D.E !

$$\left(\frac{h\omega^i}{s}\right)_{,0}=0 \Rightarrow \frac{h\omega^i}{s}=\left.\frac{h\omega^i}{s}\right|_{t_0}$$

Kelvin-Helmholz Th. $\Downarrow$

$$h\omega^i(\boldsymbol{x},t)=\sqrt{\frac{g(\boldsymbol{x},t_0)}{g(\boldsymbol{x},t)}}\,h\omega^i(\boldsymbol{x},t_0)$$

Einstein Eq.

perturbatively solved

$$g_{\mu\nu} = g_{\mu\nu}^{FLRW} + h_{\mu\nu}$$

Assume $h_{\mu\nu} \ll 1$

We have formulated

Lagrange perturbation

to enlarge

validity domain.

(longer wavelength

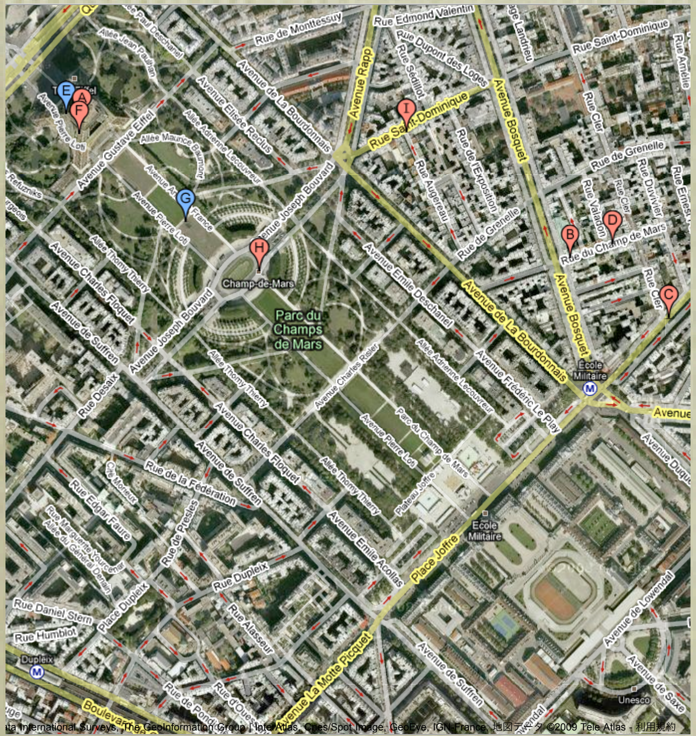
& larger amplitude)

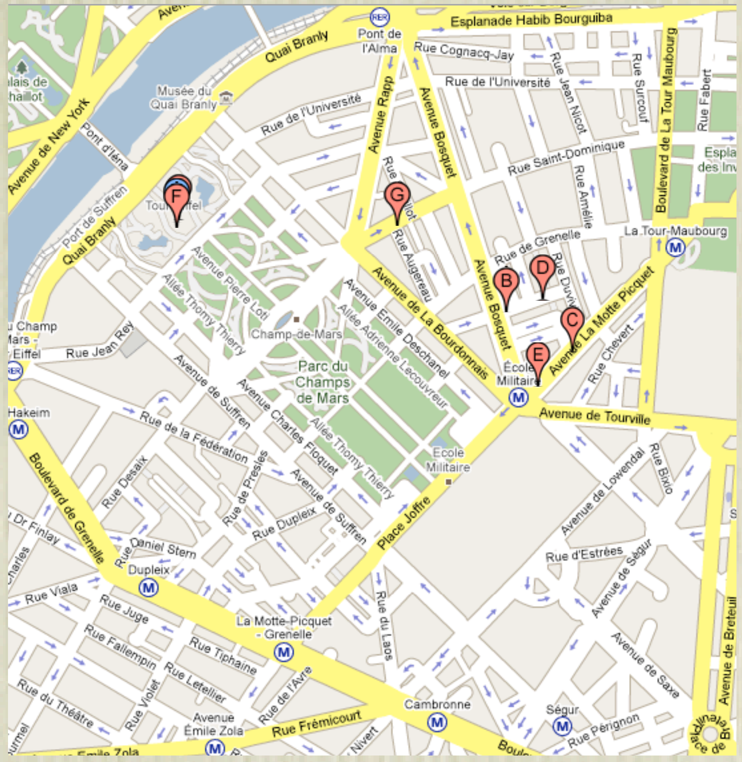
Next‥

?

≠

Evolution

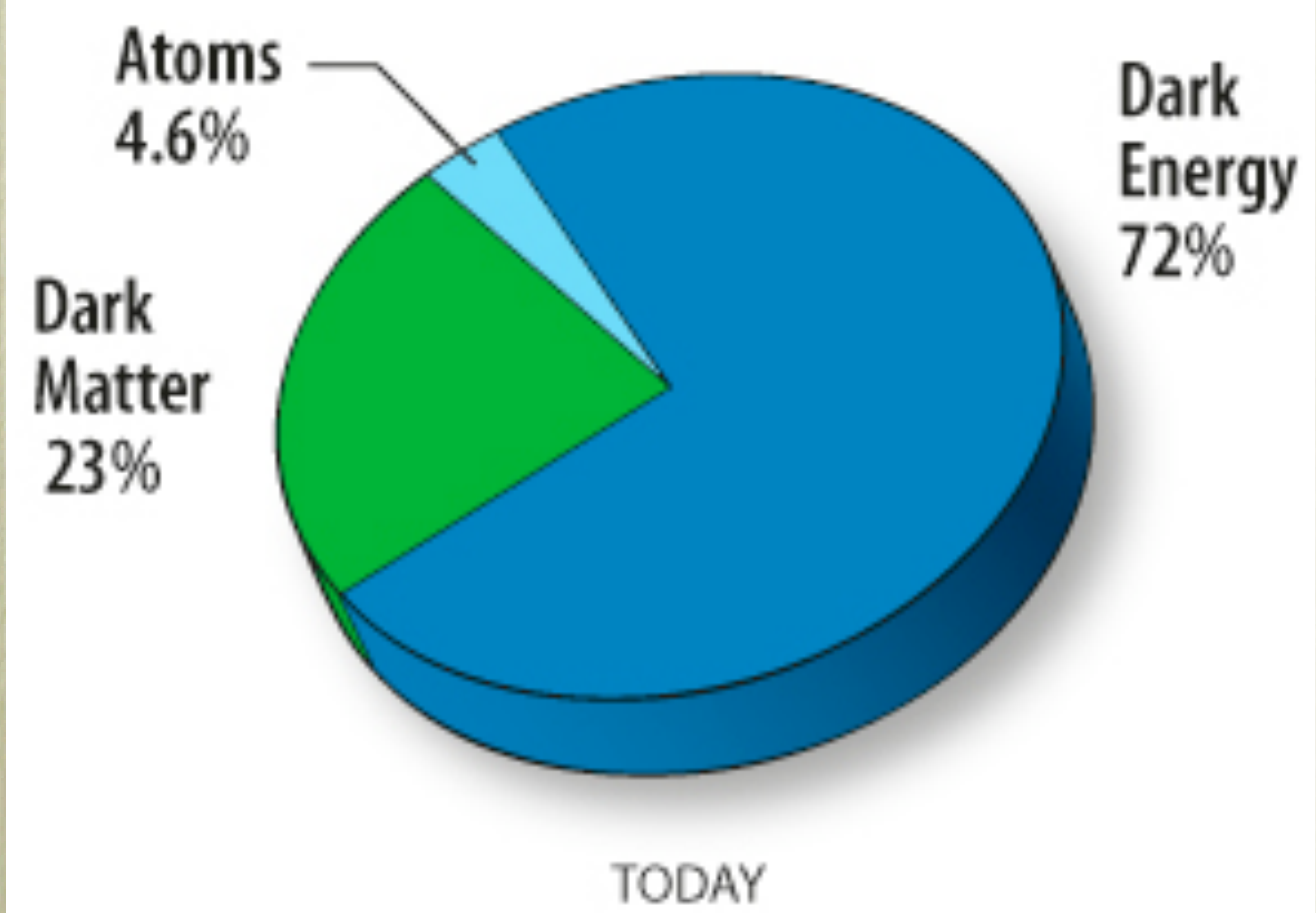




Averaging

§3 Averaging in GR cosmology

Obs. ➡ **Apparent accerelation**



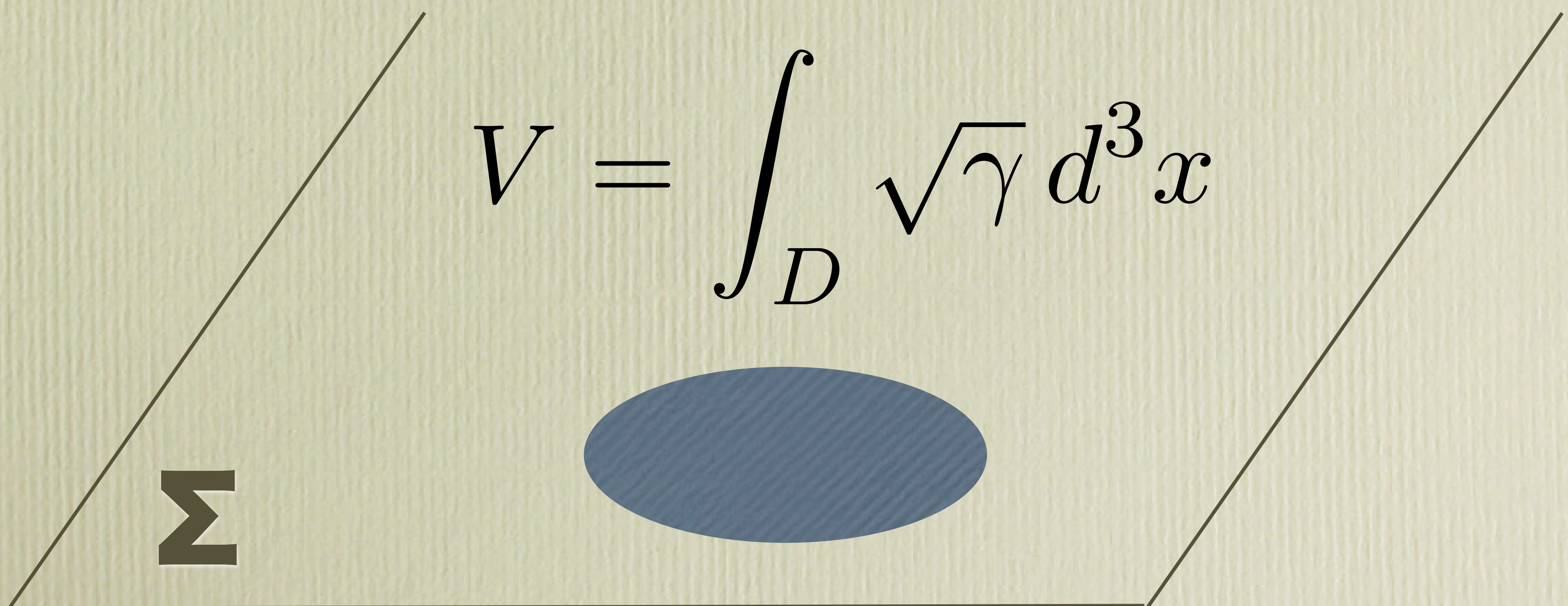
But, our universe is NOT locally homogeneuos.

Inhomogeneities may (especially in GR) give an alternative.

$$ds^2 = -(N dt)^2 + \gamma_{ij} dx^i dx^j.$$

$$N^i = 0.$$

$$V = \int_D \sqrt{\gamma}\, d^3x$$



“scale factor”

$$3\frac{\dot{a}}{a} \equiv \frac{\dot{V}}{V} = \frac{1}{V}\int_D \frac{1}{2}\gamma^{ij}\,\dot{\gamma}_{kj}\sqrt{\gamma}\,d^3x.$$

Volume expansion rate

“averaging”

$$\langle A \rangle \equiv \frac{1}{V}\int_D A\sqrt{\gamma}\,d^3x.$$

Einstein equation

$$ {}^{(3)}R + \left(K^i{}_i\right)^2 - K^i{}_j K^j{}_i = 16\pi G E, $$

$$ K^j{}_{j|i} - K^j{}_{i|j} = 8\pi G J_i, $$

$$ \dot{K}^i{}_i + N K^i{}_j K^j{}_i - N^{|i}{}_{|i} = -4\pi G N (E + S) $$

$$ E = T_{\mu\nu} n^\mu n^\nu = \frac{1}{N^2} T_{00}, $$

$$ J_i = -T_{\mu i} n^\mu = \frac{1}{N} T_{0i}, $$

$$ S = T_{ij} \gamma^{ij}. $$

Averaging the Einstein equations,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\langle N^2 E\rangle - \frac{1}{6}\langle N^2\, {}^{(3)}R\rangle - \frac{1}{6}\langle (V^i_{\ i})^2 - V^i_{\ j}V^j_{\ i}\rangle,$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\langle N^2(E+S)\rangle + \frac{1}{3}\langle (V^i_{\ i})^2 - V^i_{\ j}V^j_{\ i}\rangle + \frac{1}{3}\langle N N^{|i}_{\ \ |i} + \dot{N} K^i_{\ i}\rangle.$$

Assume

$$T^{\mu\nu} = \rho u^{\mu} u^{\nu}$$

cosmological PN metric

(1st order)

$$ds^2 = -\left(1 + 2\phi(\boldsymbol{x})\right) dt^2 + a^2 \left(1 - 2\phi(\boldsymbol{x})\right) \delta_{ij}\, dx^i dx^j$$

perturbed Einstein Eq

$$\phi_{,ii} = \frac{3}{2}\dot{a}^2\left(\frac{\rho - \rho_b}{\rho_b} + 2\phi\right),$$

$$v^i \equiv \frac{u^i}{u^0} = -\frac{2}{3a\dot{a}}\phi_{,i},$$

Require $$\dot{\bar{\rho}} + 3\frac{\dot{a}}{a}\bar{\rho} = 0$$

conservation of "cosmic mean desity"

$$\bar{\rho} \equiv \langle T_{00} \rangle + \rho_b a^2 \langle v^2 \rangle + \frac{1}{4\pi G a^2} \langle \phi_{,i} \phi_{,i} \rangle$$

$$= \langle T_{00} \rangle + \frac{5}{12\pi G a^2} \langle \phi_{,i} \phi_{,i} \rangle$$

In this case, averaged Einstein Eq

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\bar{\rho} - \frac{1}{9a^2}\langle \phi_{,i}\phi_{,i}\rangle$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\bar{\rho}.$$

Non apparent acceleration

Kasai, HA, Futamase, PTP115, 827(2006)

averaged Einstein Eq with Λ

Decreasing acceleration

Tanaka, Futamase, PTP117, 183(2007)

§4 Conclusion

1. Lagrangian perturbation in GR cosmology
2. Averaging in GR cosmology

If mean density is “properly” defined, No acceleration

Future

Extension to general cases

Application to observations

Merci !

asada@phys.hirosaki-u.ac.jp