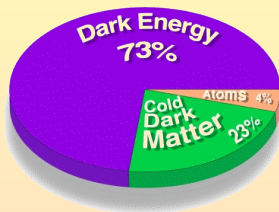# Gauss Bonnet $f(G)$ gravity.

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[1]University of Nottingham.

July 1, 2009

the cosmic pie

Wmap

- ΛCDM - fine tuning issue
- quintessence - long range forces, time dependent constants
- tachyon field - caustics
- phantom fields - unstable vacuum
- Chaplygin gas - strong ISW, loss of power in CMB
- modified gravity
  - braneworld
  - $f(R)$
  - $f(G)$

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Gauss-Bonnet

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

- topological in four dimensions
- it is one of a series of terms derived from the Euler class

$$\epsilon_{\mu_1\mu_2\mu_3\mu_4...}R^{\mu_1\mu_2} \wedge R^{\mu_3\mu_4}...$$

- α' expansion of heterotic string
- other than the Einstein tensor and the metric, the Lovelock tensor is the unique object that
  - is symmetric
  - depends on $g_{\mu\nu}$ and its first two derivatives
  - has vanishing divergence

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$f(G)$ - how hard do you have to work?

- $S = \int d^4x \sqrt{-g} \left[\frac{1}{2}R + \mathcal{L}_r + \mathcal{L}_m \right]$

Starobinsky; Carroll,deFelice,Duvvuri,Easson,Trodden and Turner;Appleby and Battye; Amendola, Charmousis, Davis; Koivisto,

Mota; Tsujikawa, Sami.

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- $\hookrightarrow S = \int d^4x \sqrt{-g} \left[f(R) + \mathcal{L}_r + \mathcal{L}_m \right]$

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dynamical systems - quintessence

$$H^2 = \frac{1}{3}[\frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_m]$$
$$\dot{H} = -\frac{1}{2}[\dot{\phi}^2 + (1 + w_m)\rho_m]$$
$$\ddot{\phi} + 3H\dot{\phi} = -dV/d\phi$$

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dynamical systems - quintessence

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$$\dot{H} = -\frac{1}{2}[\dot{\phi}^2 + (1 + w_m)\rho_m]$$

$$\ddot{\phi} + 3H\dot{\phi} = -dV/d\phi$$

$$x = \frac{\dot{\phi}}{H},$$

$$y = \frac{\sqrt{V}}{H}$$

$$\lambda = -\frac{1}{V}dV/d\phi$$

Halliwell; Wetterich; Burd and Barrow; Copeland, Liddle and Wands.
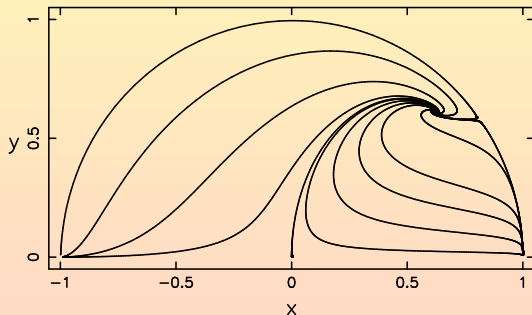
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dynamical systems - quintessence

$$dx/dN = f_1(x, y, \lambda)$$
$$dy/dN = f_2(x, y, \lambda)$$
$$d\lambda/dN = f_3(x, y, \lambda)$$

where $N = ln(a)$.

the field equations

$$
\begin{aligned}
3H^2 &= Gf_G - f - 24H^3\dot{f}_G + \rho_r + \rho_m\,, \\
-2\dot{H} &= -8H^3\dot{f}_G + 16H\dot{H}\dot{f}_G + 8H^2\ddot{f}_G + (\rho_r + p_r) + \rho_m\,, \\
\dot{\rho}_r + 4H\rho_r &= 0\,, \\
\dot{\rho}_m + 3H\rho_m &= 0\,.
\end{aligned}
$$

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dynamical system variables

$$
\begin{aligned}
x_1 &= \frac{Gf_G}{3H^2}, \\
x_2 &= -\frac{f}{3H^2}, \\
x_3 &= -8H\dot{f}_G, \\
x_4 &= \Omega_r = \frac{\rho_r}{3H^2}, \\
x_5 &= \frac{G}{24H^4} = \frac{\dot{H}}{H^2} + 1.
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\end{aligned}
$$

N.B. we may express G and H in terms of x_2 and x_5, so x_1 is not an independent variable. The Friedman equation is now

$$
\Omega_m = 1 - x_1 - x_2 - x_3 - x_4
$$

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dynamical system variables

$$\frac{dx_1}{dN} = -\frac{x_3 x_5}{m} - x_3 x_5 - 2x_1 x_5 + 2x_1 \,,$$

$$\frac{dx_2}{dN} = \frac{x_3 x_5}{m} - 2x_2 x_5 + 2x_2 \,,$$

$$\frac{dx_3}{dN} = -x_3 + 2x_5 - x_3 x_5 + 1 - 3x_1 - 3x_2 + x_4 \,,$$

$$\frac{dx_4}{dN} = -2x_4 - 2x_4 x_5 \,,$$

$$\frac{dx_5}{dN} = -\frac{x_3 x_5^2}{x_1 m} - 4x_5^2 + 4x_5 \,.$$

and introduce

$$r \equiv -\frac{Gf_G}{f} = \frac{x_1}{x_2} \,, \qquad m \equiv \frac{Gf_{GG}}{f_G} \,.$$

Gauss Bonnet $f(G)$ gravity.

dynamical system variables

This is now an autonomous system, once $f(G)$ is given.

$$f(G) \Rightarrow G\left(r = \frac{x_1}{x_2}\right) \Rightarrow m\left(\frac{x_1}{x_2}\right)$$

swap $f(G)$ for $m(r)$.
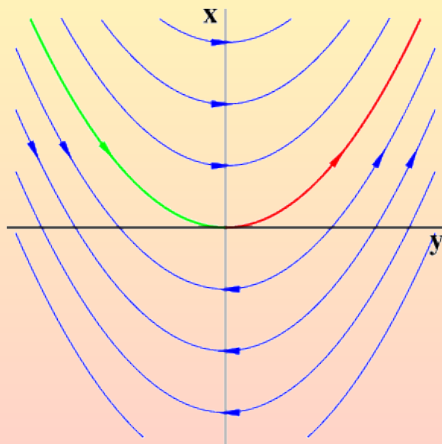
$$m(r) \longrightarrow f(G)$$

dynamical system equilibrium manifolds
critical points and critical lines

$$\frac{dx}{dt} = xy, \qquad \frac{dy}{dt} = x$$

has a critical *line* at $x = 0$, and
bifurcates *without any parameters.*

s*tability* : $\begin{cases} \text{s}table \text{ } for x < 0 \\ \text{u}nstable \text{ } for x < 0 \end{cases}$

our dynamical system

- $L_1 : (1 - \lambda, \lambda, 0, 0, 1)$.
  $\Omega_m = 0$, $\Omega_r = 0$, $\Omega_{DE} = 1$, $w_{DE} = -1$, $w_{eff} = -1$.

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- $L_2 : \left(\frac{1}{6}\lambda, -\frac{1}{3}\lambda, \lambda, 0, -\frac{1}{2}\right)$, $m = -\frac{1}{2}$.
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  $\Omega_m = 1 - \frac{5}{6}\lambda,\ \Omega_r = 0,\ \Omega_{DE} = \frac{5}{6}\lambda,\ w_{DE} = 0,\ w_{eff} = 0$.
- $L_3 : \left(\frac{\lambda}{\lambda-2}, -\frac{2\lambda}{\lambda-2}, \frac{2\lambda-2}{\lambda-2}, 0, \lambda\right),\ m = -\frac{1}{2}$.
  $\Omega_m = \Omega_r = 0,\ \Omega_{DE} = 1,\ w_{DE} = -\frac{2}{3}\lambda - \frac{1}{3},\ w_{eff} = -\frac{2}{3}\lambda - \frac{1}{3}$.

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our dynamical system

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- $L_4 : (\frac{1}{4}\lambda, -\frac{1}{2}\lambda, \lambda, 1 - \frac{3}{4}\lambda, -1)$, $m = -\frac{1}{2}$.
  $\Omega_m = 0$, $\Omega_r = 1 - \frac{3}{4}\lambda$, $\Omega_{DE} = \frac{3}{4}\lambda$, $w_{DE} = \frac{1}{3}$, $w_{eff} = \frac{1}{3}$.
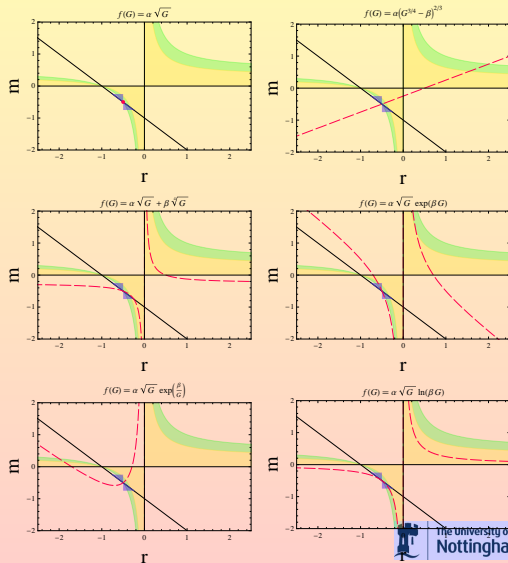
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our dynamical system

line	property	remarks
L1	deSitter	
L2	matter scaling	$m = -\frac{1}{2}$, $r = -\frac{1}{2}$
L3	dark energy dominated	$m = -\frac{1}{2}$, $r = -\frac{1}{2}$
L4	radiation scaling	$m = -\frac{1}{2}$, $r = -\frac{1}{2}$

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our dynamical system

$$r \equiv -\frac{G f_G}{f}$$

$$= \frac{x_1}{x_2},$$

$$m \equiv \frac{G f_{GG}}{f_G}$$

we also have

$$\frac{dr}{dN} = r(m + r + 1)\frac{\dot{G}}{HG}$$

our dynamical system

line	property	remarks
L1	deSitter	
L2	matter scaling	$m = -\frac{1}{2}, r = -\frac{1}{2}$
L3	dark energy dominated	$m = -\frac{1}{2}, r = -\frac{1}{2}$
L4	radiation scaling	$m = -\frac{1}{2}, r = -\frac{1}{2}$

look for solutions of the form
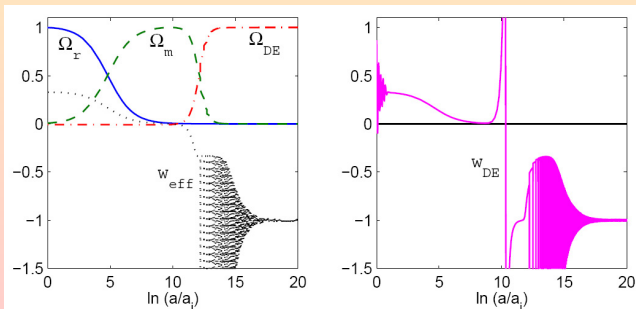
$$L4 \longrightarrow L2 \longrightarrow L1$$

recall: $r \equiv -\frac{Gf_G}{f} = \frac{x_1}{x_2}$, $\qquad m \equiv \frac{Gf_{GG}}{f_G}$.

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an example

$$f(G) = \alpha(G^p - \beta)^q$$

- matter point exists $\Rightarrow 2pq = 1$.
- matter point unstable $\Rightarrow q > 0$.

$$consider \ f(G) = \alpha(G^{3/4} - \beta)^{2/3}$$

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Summary

- $f(G)$ models provide a rich arena for studying cosmological models
- $f(G)$ may be usefully cast as a dynamical system

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the field equations

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3H^2 &= Gf_G - f - 24H^3\dot{f}_G + \rho_r + \rho_m\,, \\
-2\dot{H} &= -8H^3\dot{f}_G + 16H\dot{H}\dot{f}_G + 8H^2\ddot{f}_G + (\rho_r + p_r) + \rho_m\,, \\
\dot{\rho_r} + 4H\rho_r &= 0\,, \\
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$$

using $3H^2 = \rho$, $-2\dot{H} = \rho + p$ we define

$$
\begin{aligned}
\rho_{DE} &= Gf_G - f - 24H^3\dot{f}_G\,, \\
p_{DE} &= 16H^3\dot{f}_G + 16H\dot{H}\dot{f}_G + 8H^2\ddot{f}_G - Gf_G + f\,.
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\end{aligned}$$

$$\dot{\rho_{DE}} + 3H\rho_{DE} = 0\,.$$

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dynamical system equilibrium manifolds

critical points and critical lines

$$\frac{dx}{dt} = -x, \qquad \frac{dy}{dt} = y^2 - \beta$$

has a critical *points* at
$x = 0$, $y = \pm\sqrt{\beta}$, for $\beta > 0$.