Towards an understanding of the Cosmology of f(R) gravity

Peter Dunsby
University of Cape Town

The purple Disa....almost as elusive as Dark energy!

Invisible Universe, Paris , 1 July 2009

1+3 approach provides a powerful toolbox .....

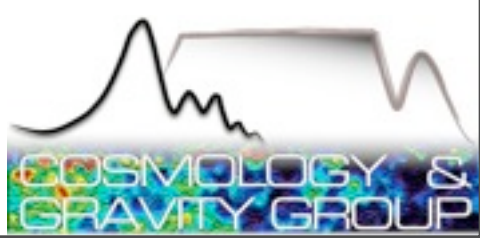
With Sante Carloni, Antonio Troisi, Salvatore Capozziello, Kishore Ananda , Julien Larena, Naureen Goheer, Rituparno Goswami, Mohamed Abdelwahab, Amare Abebe, Anne-Marie Nkioki,

Some notation

- From the time-like flow u^a we construct the projection onto surfaces orthogonal to the flow: $h_{ab} = g_{ab} + u_a u_b$.
- Three-volume form: $\varepsilon_{abc} = \eta_{abcd} u^d$
- Covariant convective derivative on scalar: $\dot{f} = u^a \nabla_a f$.
- Spatial covariant derivative: $\tilde{\nabla}_a f = h^b{}_a \nabla_b f$.
- Kinematics of u^a gives geometry of congruence of flow lines.

acceleration expansion shear vorticity

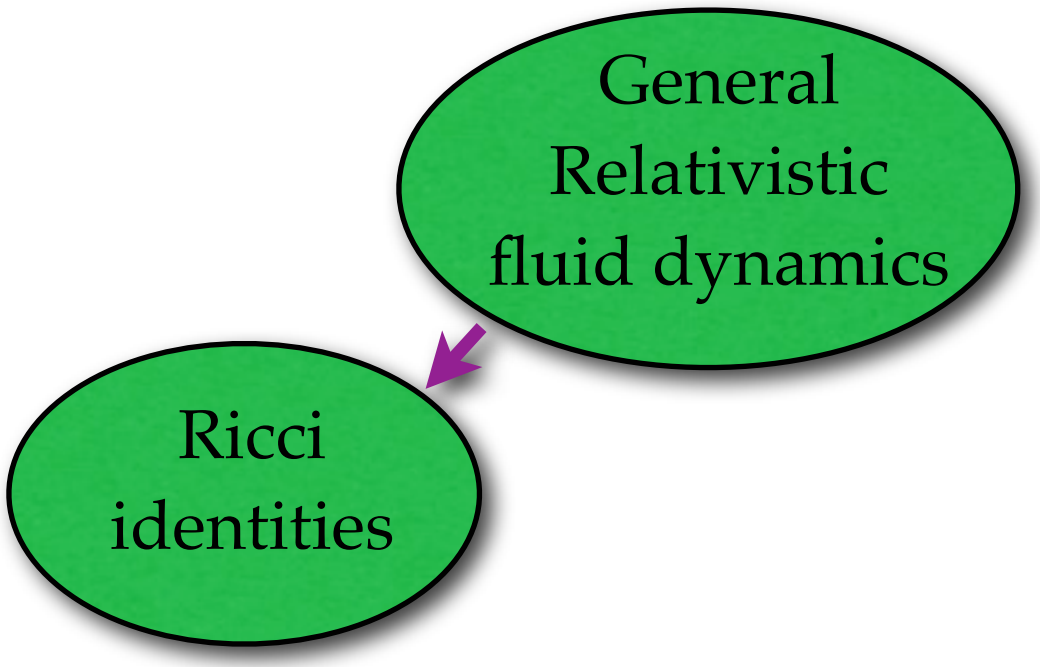
$$\nabla_a u_b = -u_a \dot{u}_b + \tfrac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab}$$



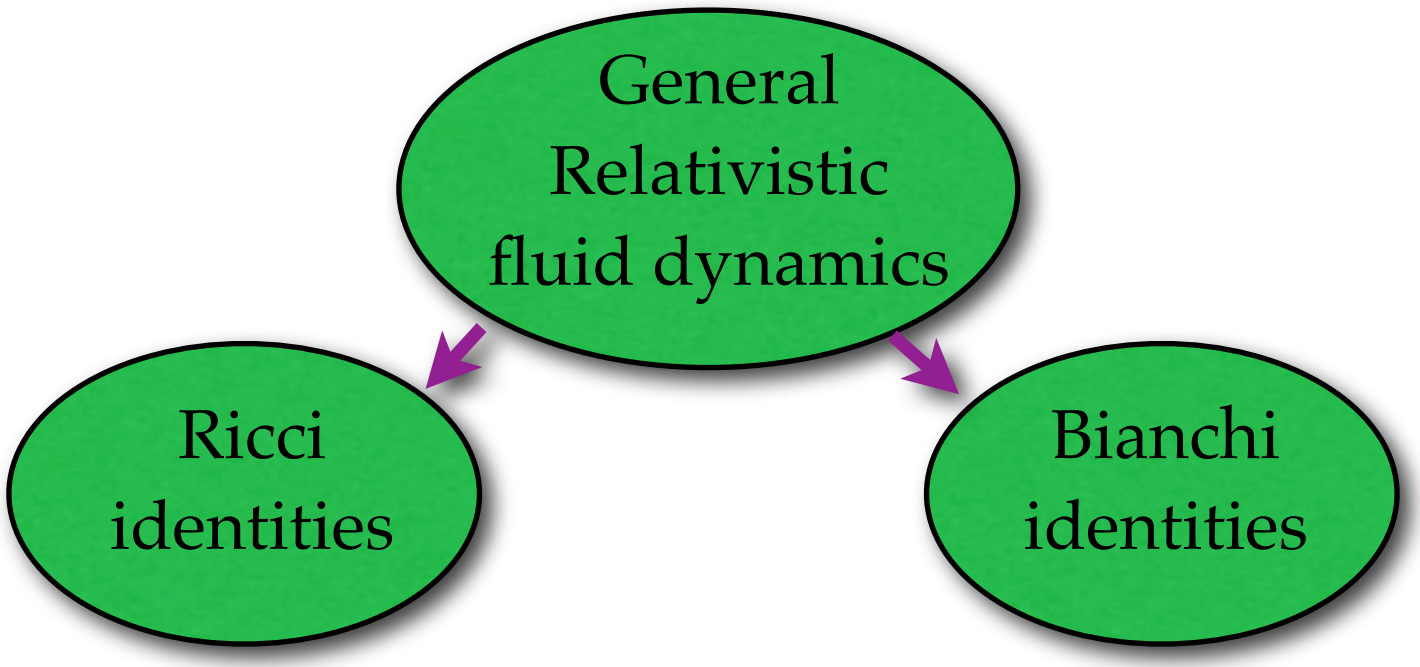
Structure of the 1+3 equations

General
Relativistic
fluid dynamics

Structure of the 1+3 equations



Structure of the 1+3 equations



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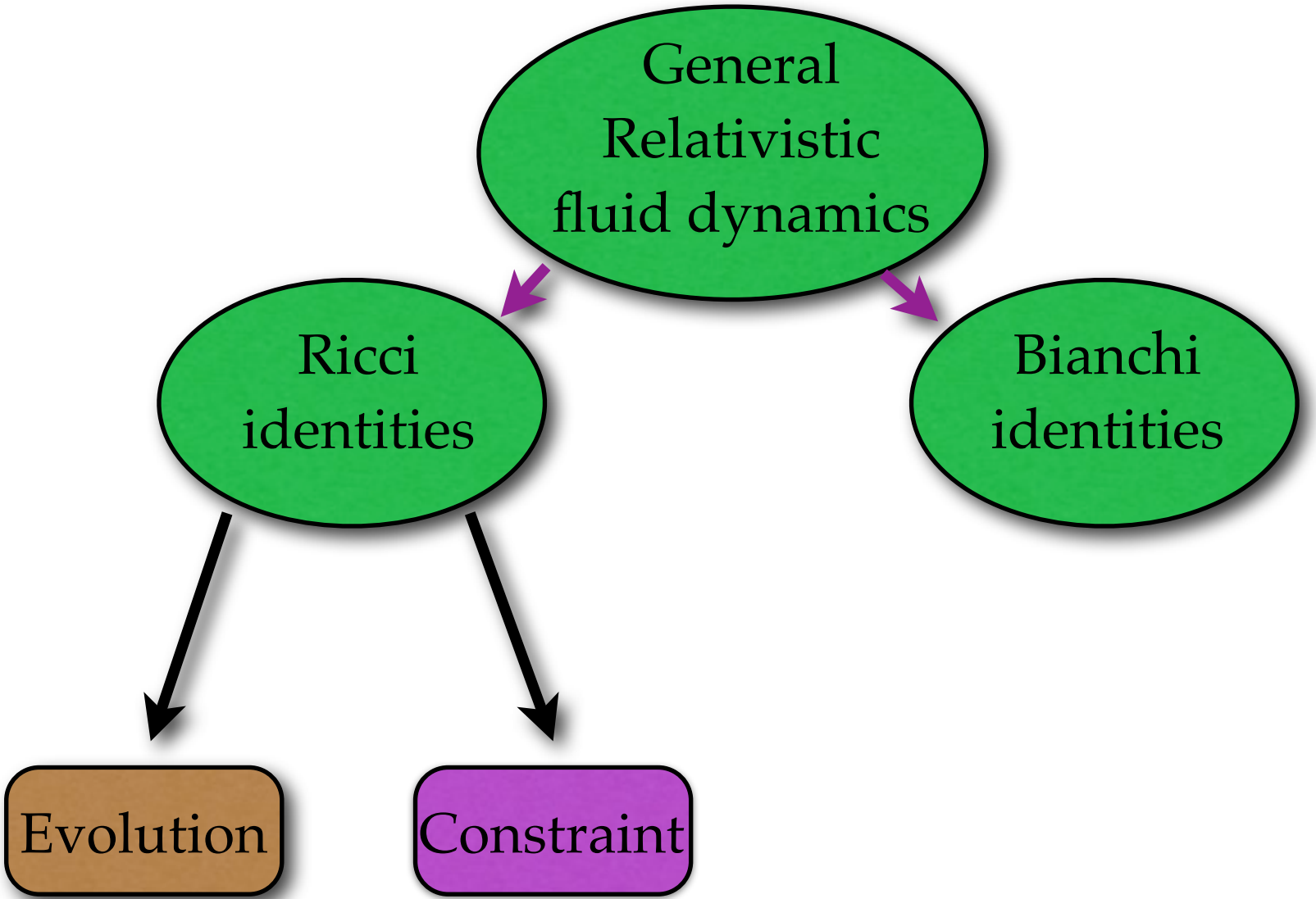

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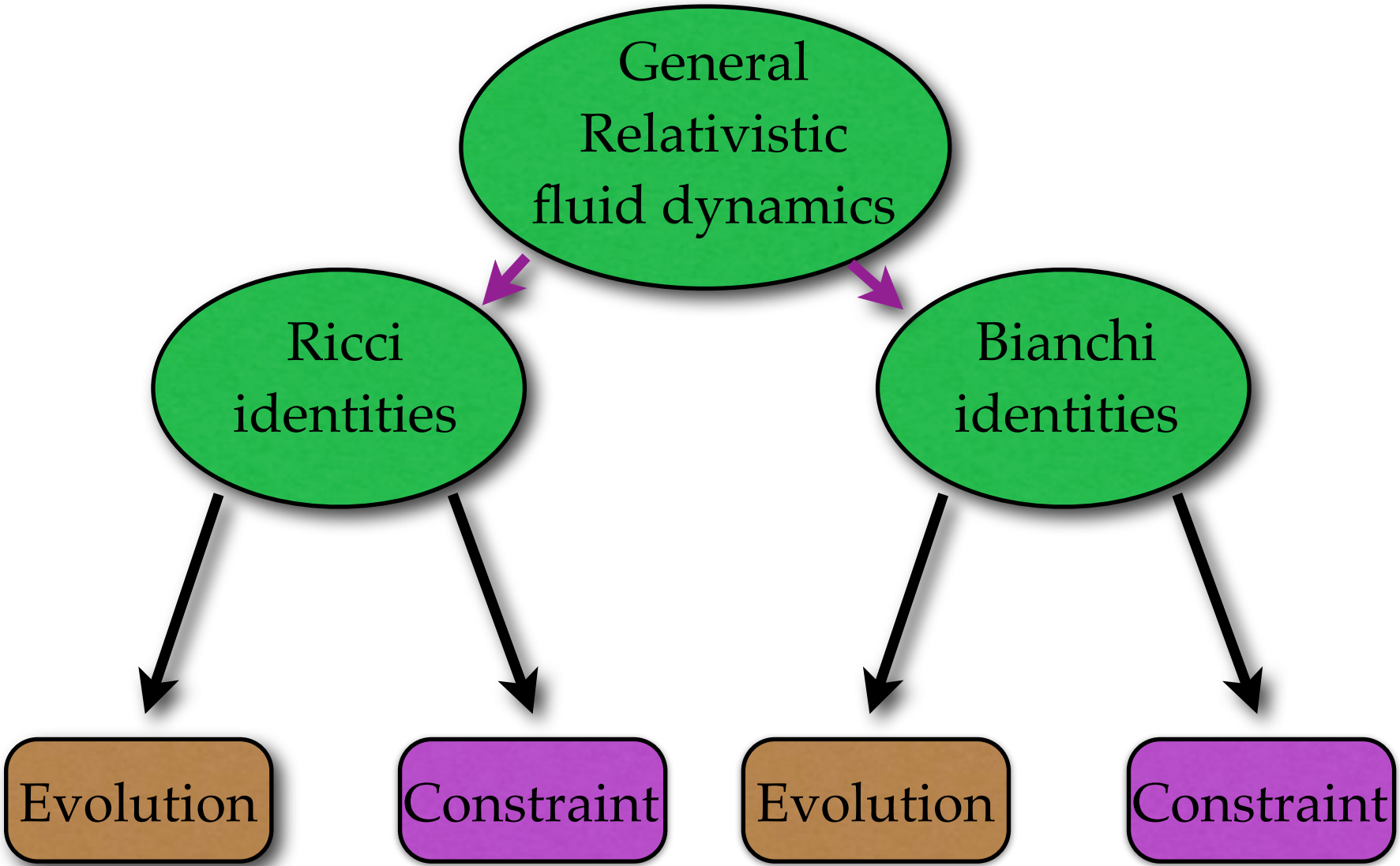

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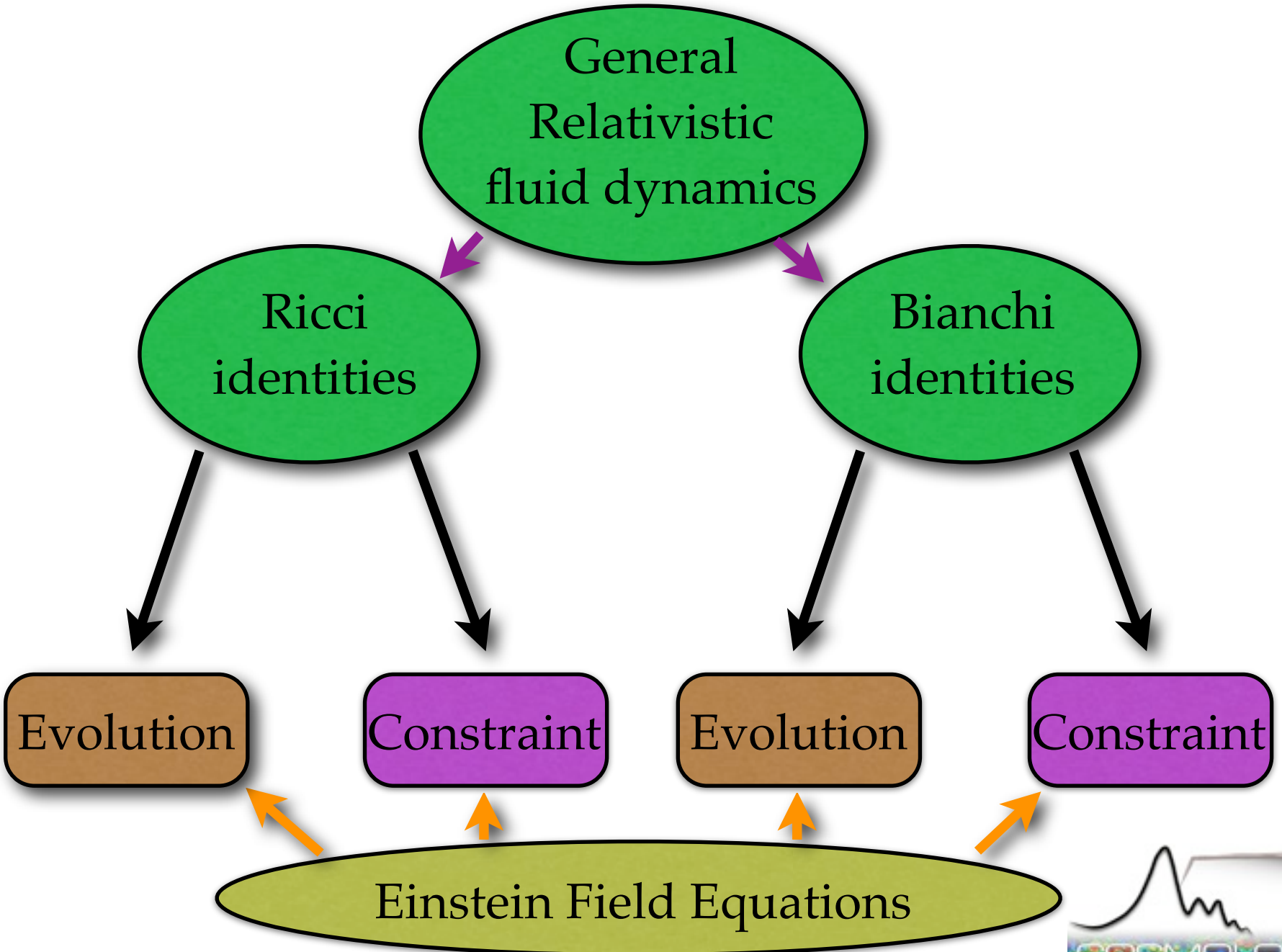

Fourth order gravity

The class of models we will consider can be derived from the classical action:

$$\mathcal{A} = \int d^4x\sqrt{-g}\left[f(R) + \mathcal{L}_m\right] ,$$

Varying the action with respect to the metric gives the following field equations:

$$f'G_{ab} = f'\left(R_{ab} - \frac{1}{2}\,g_{ab}R\right) = T^m_{ab} + \frac{1}{2}g_{ab}\left(R - Rf'\right) + \nabla_b\nabla_a f' - g_{ab}\nabla_c\nabla^c f' ,$$

$$G_{ab} = \tilde{T}^m_{ab} + T^R_{ab} = T^{tot}_{ab} ,$$

This last step is extremely important as it allows us to treat 4th order gravity as standard GR in the presence of two effective fluids. It is this that makes using the covariant approach particularly straightforward.

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This last step is extremely important as it allows us to treat 4th order gravity as standard GR in the presence of two effective fluids. It is this that makes using the covariant approach particularly straightforward.

The energy-momentum tensor of the curvature "fluid" can be decomposed as follows:

$$\mu^R = \frac{1}{f'}\left[\frac{1}{2}(Rf' - f) - \Theta f''\dot{R} + f''\tilde{\nabla}^2 R + f''\,\dot{u}_b\tilde{\nabla}R\right] ,$$

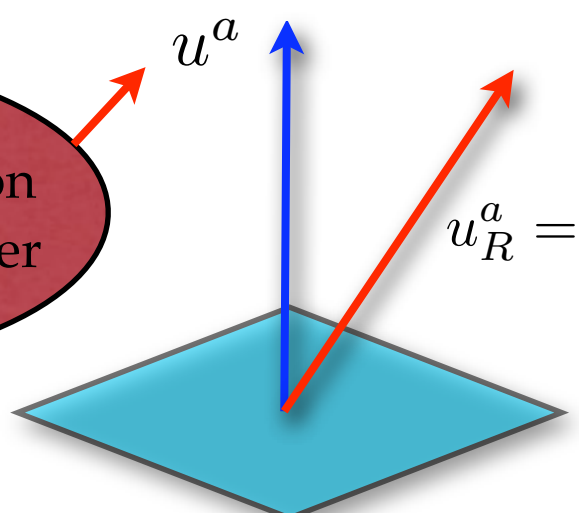
$$p^R = \frac{1}{f'}\left[\frac{1}{2}(f - Rf') + f''\ddot{R} + 3f'''\dot{R}^2 + \frac{2}{3}\Theta f''\dot{R} - \frac{2}{3}f''\tilde{\nabla}^2 R + \right.$$
$$\left. -\frac{2}{3}f'''\tilde{\nabla}^a R\tilde{\nabla}_a R - \frac{1}{3}f''\,\dot{u}_b\tilde{\nabla}R\right] ,$$

Note no background contribution.

$$q_a^R = -\frac{1}{f'}\left[f'''\dot{R}\tilde{\nabla}_a R + f''\tilde{\nabla}_a\dot{R} - \frac{1}{3}f''\tilde{\nabla}_a R\right] ,$$

$$\pi_{ab}^R = \frac{1}{f'}\left[f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + f'''\tilde{\nabla}_{\langle a}R\tilde{\nabla}_{b\rangle}R + \sigma_{ab}\dot{R}\right] .$$

Taken to be the motion of STANDARD matter

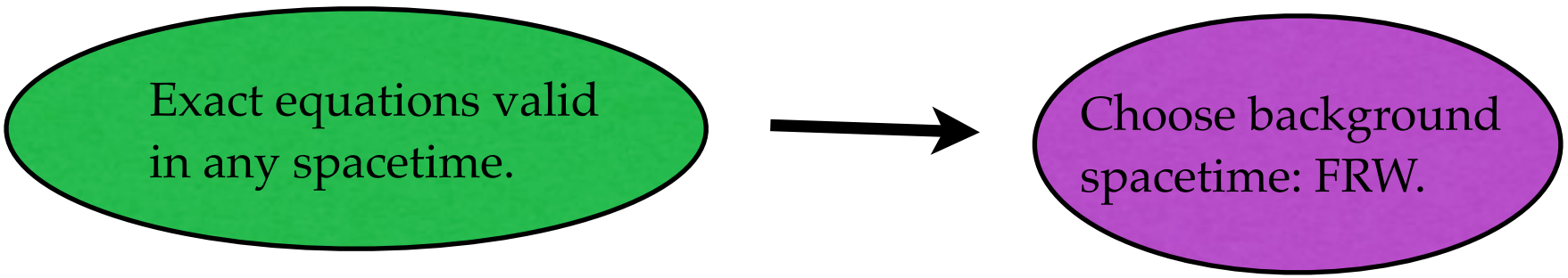


So one can think of this as a curvature "fluid" moving relative to u^a

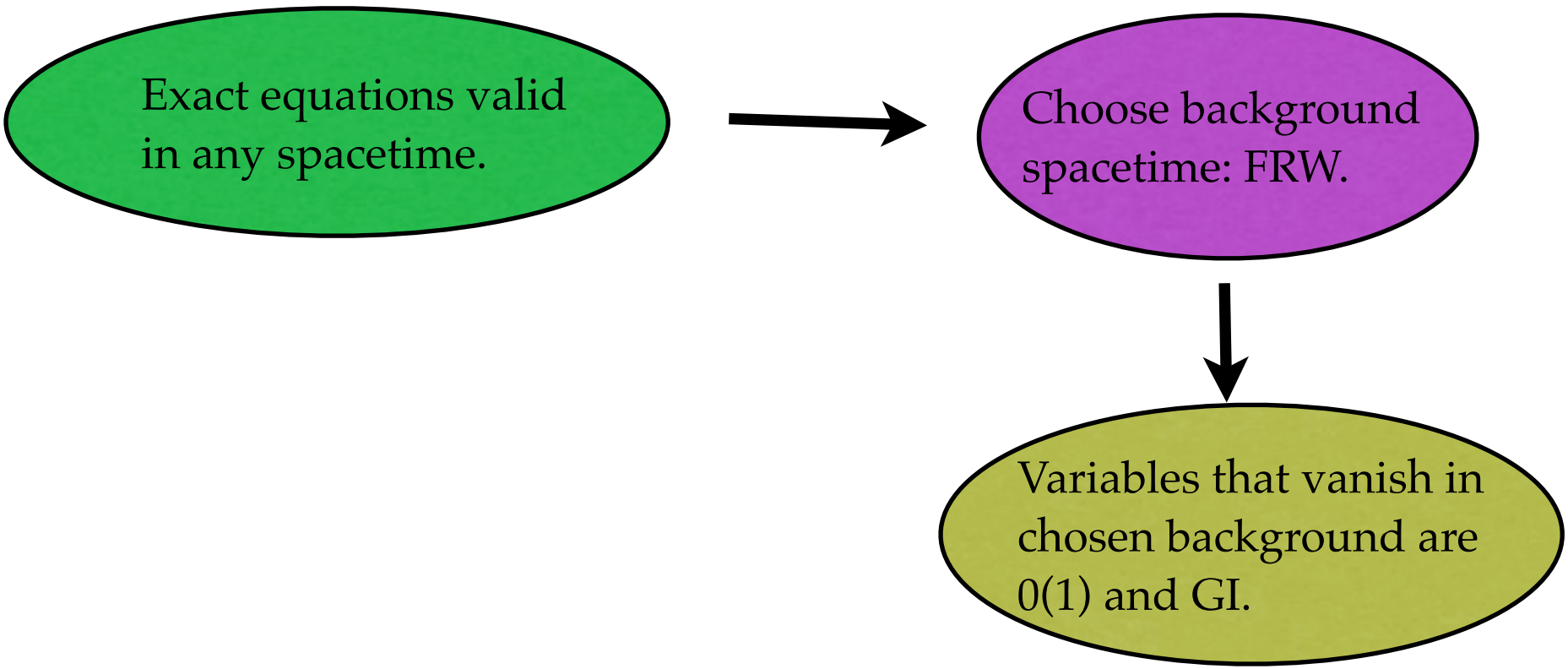
Linearisation

Exact equations valid
in any spacetime.

Linearisation



Linearisation



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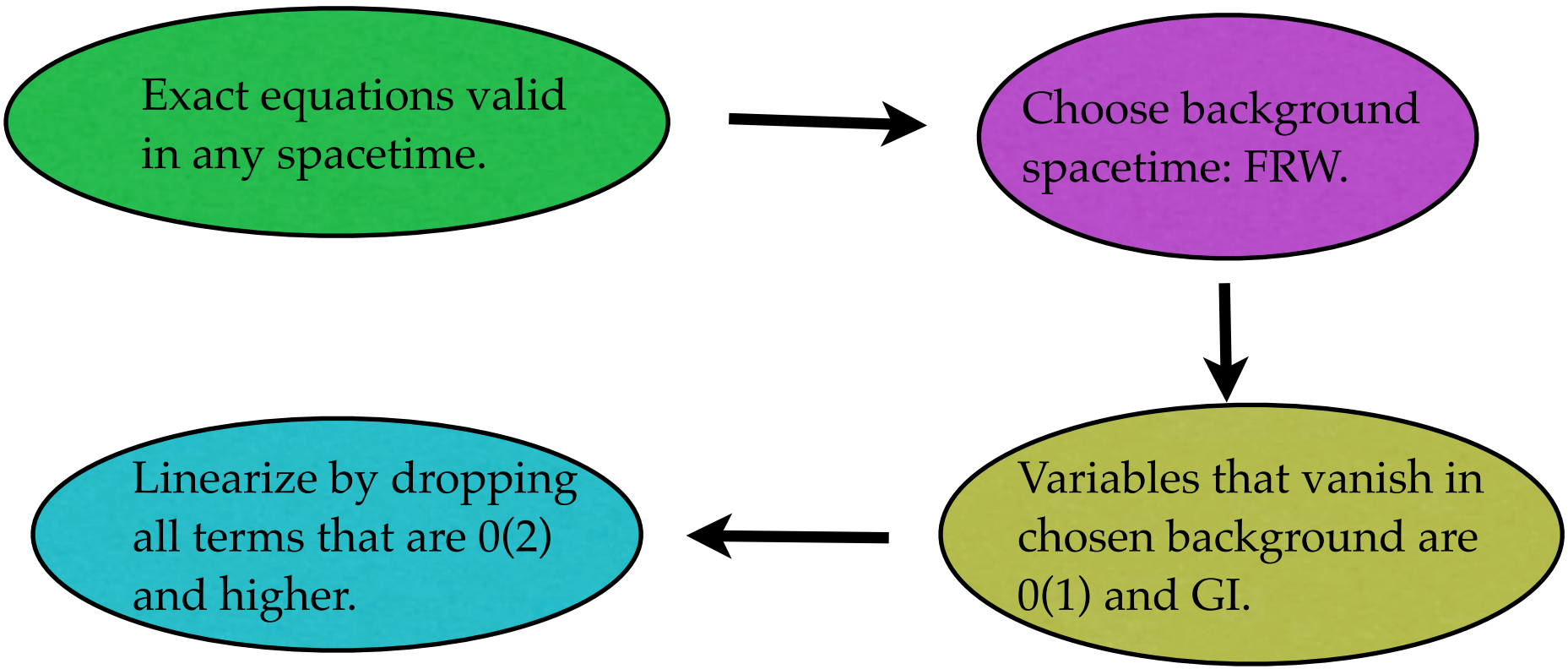

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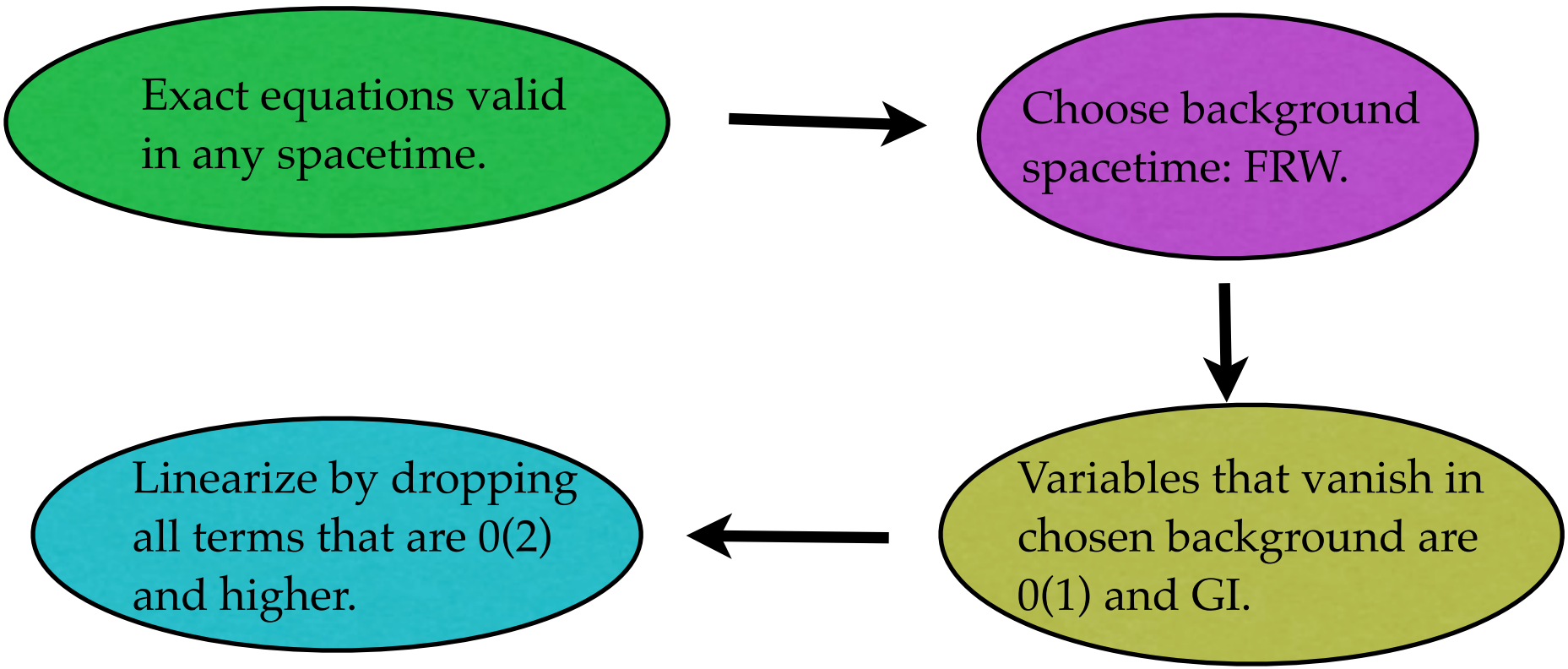
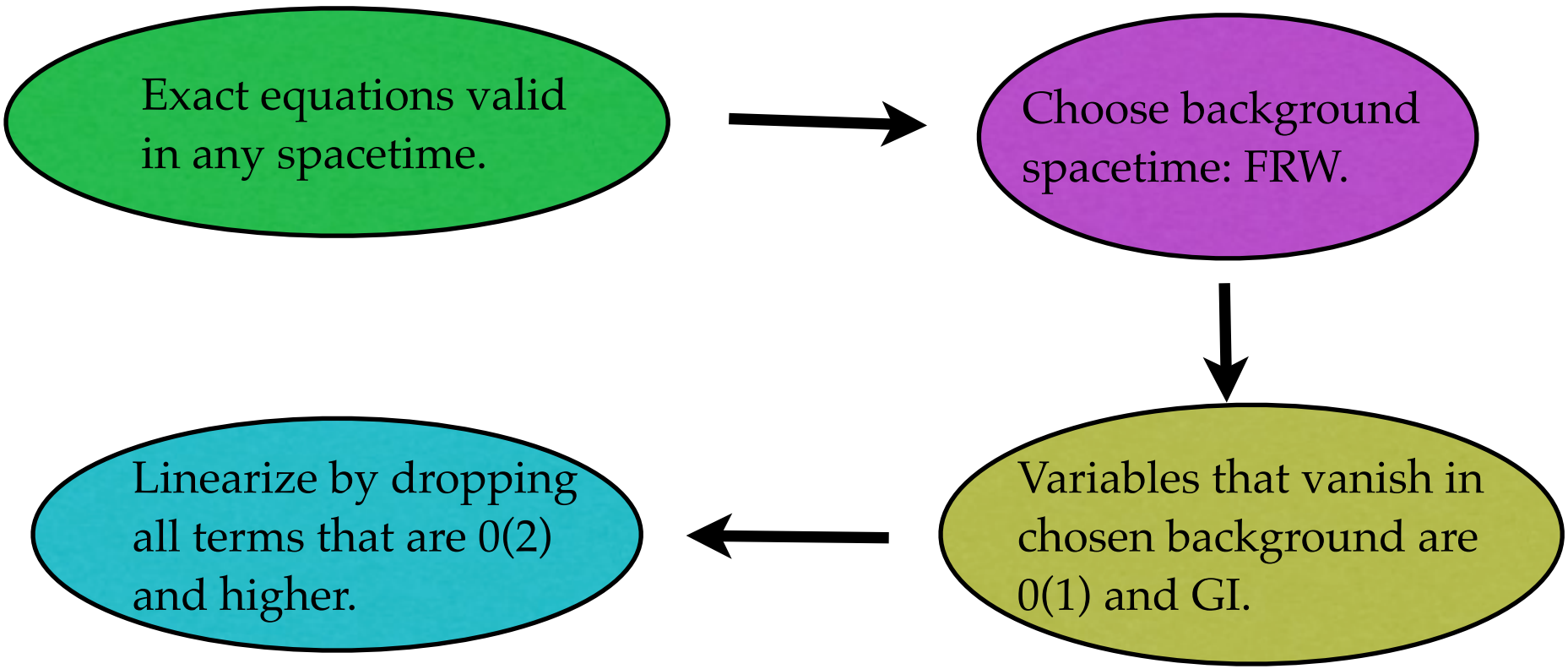


$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 + \sigma_{ab}\sigma^{ab} - 2\omega_a\omega^a - \tilde{\nabla}^a\dot{u}_a + \dot{u}_a\dot{u}^a + \tfrac{1}{2}(\mu^{tot} + 3p^{tot}) = 0$$

Linearisation



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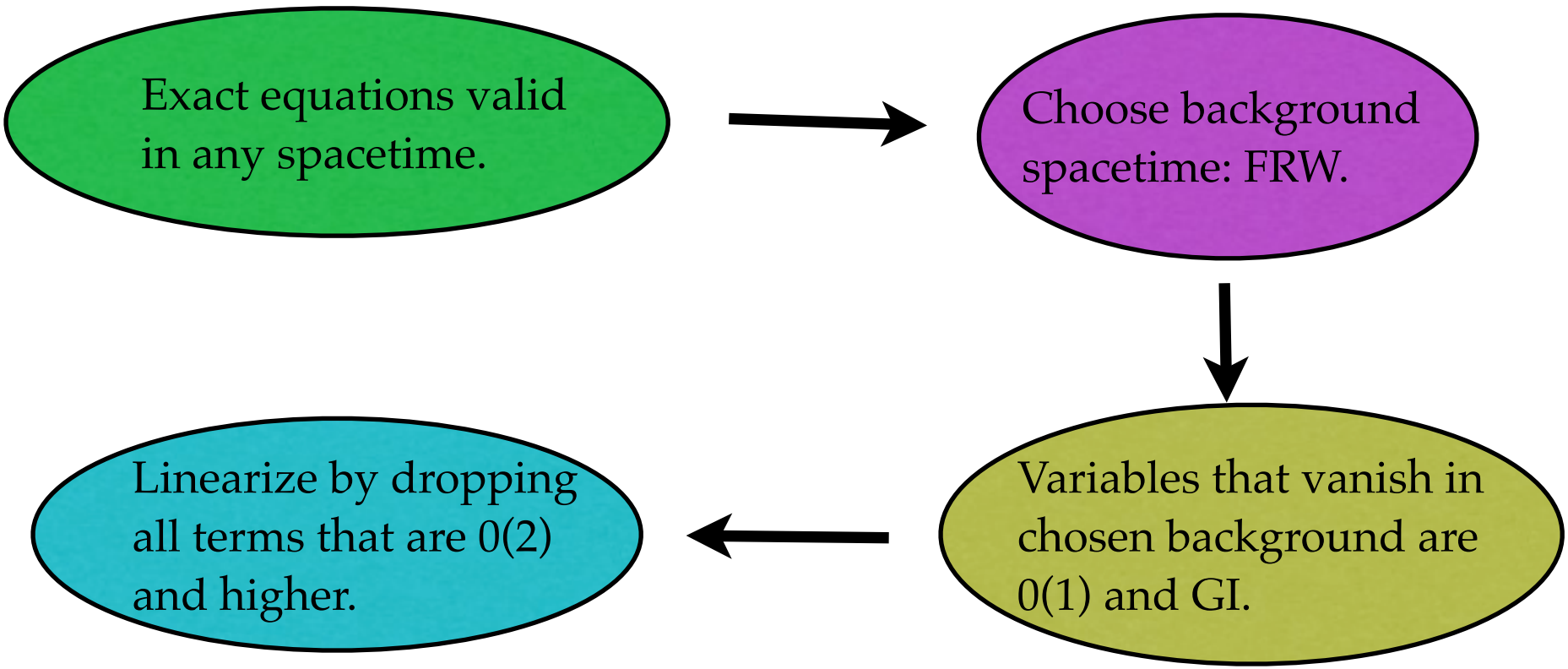

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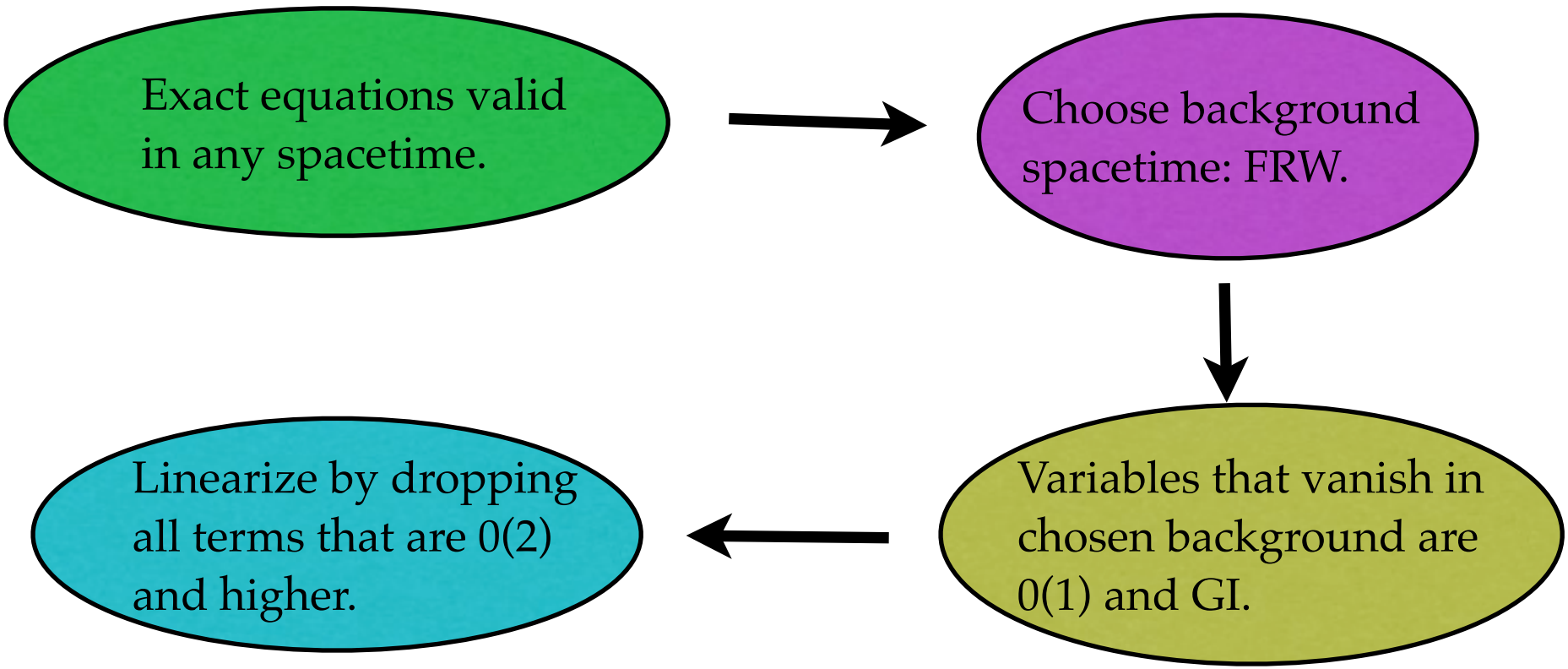


$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 + \sigma_{ab}\sigma^{ab} - 2\omega_a\omega^a - \tilde{\nabla}^a \dot{u}_a + \dot{u}_a\dot{u}^a + \tfrac{1}{2}(\mu^{tot} + 3p^{tot}) = 0$$

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 - \tilde{\nabla}^a \dot{u}_a + \tfrac{1}{2}(\mu^{tot} + 3p^{tot}) = 0 \longrightarrow$$

Almost FRW model.

The linear gravitational equations

Propagation

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 - \tilde{\nabla}^a A_a + \tfrac{1}{2}(\tilde{\mu}^m + 3\tilde{p}^m) = -\tfrac{1}{2}(\mu^R + 3p^R),$$

$$\dot{\omega}_a + 2H\omega_a + \tfrac{1}{2}\mathrm{curl}\, A_a = 0\,,$$

$$\dot{\sigma}_{ab} + 2H\sigma_{ab} + E_{ab} - \tilde{\nabla}_{\langle a} A_{b\rangle} = -q^R_a,$$

$$\dot{E}_{ab} + 3HE_{ab} - \mathrm{curl}\, H_{ab} + \tfrac{1}{2}(\tilde{\mu}^m + \tilde{p}^m)\sigma_{ab} = -\tfrac{1}{2}(\mu^R + p^R)\sigma_{ab} - \tfrac{1}{2}\dot{\pi}^R_{\langle ab\rangle} - \tfrac{1}{2}\tilde{\nabla}_{\langle a} q^R_{b\rangle} - \tfrac{1}{6}\Theta\pi^R_{ab},$$

$$\dot{H}_{ab} + 3HH_{ab} + \mathrm{curl}\, E_{ab} = \tfrac{1}{2}\mathrm{curl}\,\pi^R_{ab}\,,$$

Constraint

$$\tilde{\nabla}^b\sigma_{ab} - \mathrm{curl}\,\omega_a - \tfrac{2}{3}\tilde{\nabla}_a\Theta = -q^R_a,$$

$$\mathrm{curl}\,\sigma_{ab} + \tilde{\nabla}_{\langle a}\omega_{b\rangle} - H_{ab} = 0\,,$$

$$\tilde{\nabla}^b E_{ab} - \tfrac{1}{3}\tilde{\nabla}_a\tilde{\mu}^m = -\tfrac{1}{2}\tilde{\nabla}^b\pi^R_{ab} + \tfrac{1}{3}\tilde{\nabla}_a\mu^R - \tfrac{1}{3}\Theta q^R_a,$$

$$\tilde{\nabla}^b H_{ab} - (\tilde{\mu}^m + \tilde{p}^m)\omega_a = -\tfrac{1}{2}\mathrm{curl}\, q^R_a + (\mu^R + p^R)\omega_a,$$

$$\tilde{\nabla}^a\omega_a = 0\,,$$

The linear conservation equations

The Bianchi identities:

$$\begin{cases} \tilde{T}^{M;b}_{ab} = \dfrac{T^{m;b}_{ab}}{f'} - \dfrac{f''}{f'^2}\, T^{m}_{ab}\, R^{;b} \,, \\ T^{R;b}_{ab} = \dfrac{f''}{f'^2}\, \tilde{T}^{M}_{ab}\, R^{;b} \,, \end{cases}$$

Matter

$$\begin{cases} \dot{\mu}^m = -\,\Theta\,(\mu^m + p^m) \,, \\ \tilde{\nabla}^a p^m = -(\mu^m + p^m)\,\dot{u}^a \,, \end{cases}$$

Curvature

$$\begin{cases} \dot{\mu^R} + \tilde{\nabla}^a q^R_a = -\,\Theta\,(\mu^R + p^R) + \mu^m \dfrac{f''\,\dot{R}}{f'^2} \,, \\ \dot{q}^R_{\langle a\rangle} + \tilde{\nabla}_a p^R + \tilde{\nabla}^b \pi^R_{ab} = -\,\frac{4}{3}\,\Theta\, q^R_a - (\mu^R + p^R)\,\dot{u}_a + \mu^m \dfrac{f''\,\tilde{\nabla}_a R}{f'^2} \,, \end{cases}$$

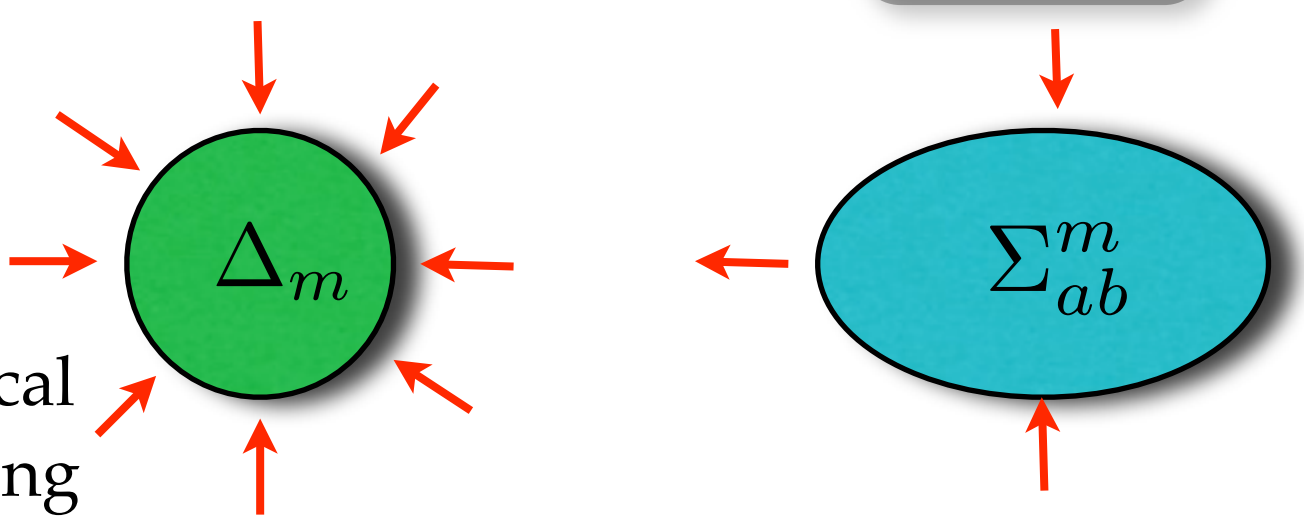
Perturbation variables

The natural set of inhomogeneity variables are:

$$\Delta_a^m = \frac{S^2}{\mu^m}\tilde{\nabla}^2\mu^m\,, \qquad Z = S^2\tilde{\nabla}^2\Theta\,, \qquad C = S^3\tilde{\nabla}^2\tilde{R}\,,$$

together with: $\mathcal{R} = S^2\tilde{\nabla}^2 R\,, \qquad \Re = S^2\tilde{\nabla}_a\dot{R}\,,$

$$S\tilde{\nabla}_a\mathcal{D}_b^m = \Sigma_{ab}^m + W_{ab}^m + \frac{1}{3}\Delta_m h_{ab}$$



Spherical clumping

Pancake formation

Perturbation equations

Scalar perturbations governed by the 4th order system:

$$\dot{\Delta}_m = w\Theta\Delta_m - (1+w)Z\,,$$

$$\dot{Z} = \left(\frac{\dot{R}f''}{f'} - \frac{2\Theta}{3}\right)Z + \left[\frac{3(w-1)(3w+2)}{6(w+1)}\frac{\mu}{f'} + \frac{2w\Theta^2 + 3w(\mu^R + 3p^R)}{6(w+1)}\right]\Delta_m + \frac{\Theta f''}{f'}\Re$$

$$+\left[\frac{1}{2} - \frac{1}{2}\frac{f}{f'}\frac{f''}{f'} - \frac{f''}{f'}\frac{\mu}{f'} + \dot{R}\Theta\left(\frac{f''}{f'}\right)^2 + \dot{R}\Theta\frac{f^{(3)}}{f'}\right]\mathcal{R} - \frac{w}{w+1}\tilde{\nabla}^2\Delta_m - \frac{f''}{f'}\tilde{\nabla}^2\mathcal{R}\,,$$

$$\dot{\mathcal{R}} = \Re - \frac{w}{w+1}\dot{R}\,\Delta_m\,,$$

$$\dot{\Re} = -\left(\Theta + 2\dot{R}\frac{f^{(3)}}{f''}\right)\Re - \dot{R}Z - \left[\frac{(3w-1)}{3}\frac{\mu}{f''} + 3\frac{w}{w+1}(p^R + \mu^R)\frac{f'}{f''} + \frac{w}{3(w+1)}\dot{R}\left(\Theta - 3\dot{R}\frac{f^{(3)}}{f''}\right)\right]\Delta_m$$

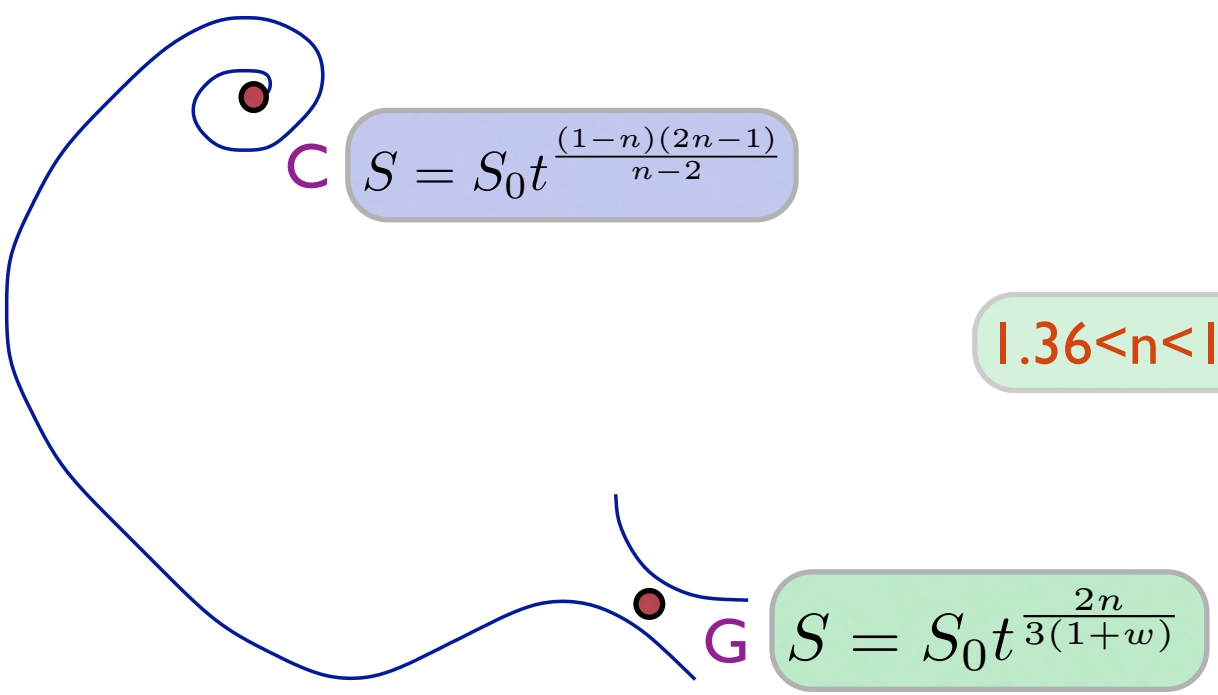
$$+\left[2\frac{K}{S^2} - \left(\frac{1}{3}\frac{f'}{f''} + \frac{f^{(4)}}{f'}\dot{R}^2 + \Theta\frac{f^{(3)}}{f'}\dot{R} - \frac{2}{9}\Theta^2 + \frac{1}{3}(\mu^R + 3p^R) + \ddot{R}\frac{f^{(3)}}{f''} - \frac{1}{6}\frac{f}{f'} + \frac{1}{2}(w+1)\frac{\mu}{f'} - \frac{1}{3}\dot{R}\Theta\frac{f''}{f'}\right)\right]\mathcal{R} + \tilde{\nabla}^2\mathcal{R}\,,$$

$$\frac{C}{S^2} + \left(\frac{4}{3}\Theta + \frac{2\dot{R}f''}{f'}\right)Z - 2\frac{\mu}{f'}\Delta_m + \left[2\dot{R}\Theta\frac{f^{(3)}}{f'} - \frac{f''}{f'}\left(f - 2\mu + 2\dot{R}\Theta f''\right)\right]\mathcal{R} + \frac{2\Theta f''}{f'}\Re - \frac{2f''}{f'}\tilde{\nabla}^2\mathcal{R} = 0\,.$$

But how do we solve these equations.....

A simple example: R^n gravity

Point	Coordinates (x, y, z)	Scale Factor
$\mathcal{A}$	$[0, 0, 0]$	$a = a_0(t - t_0)$
$\mathcal{B}$	$[-1, 0, 0]$	$a = a_0(t - t_0)^{1/2}$ (only for $n = 3/2$)
$\mathcal{C}$	$\left[\frac{2(n-2)}{2n-1}, \frac{4n-5}{2n-1}, 0\right]$	$a = a_0\, t^{\frac{(1-n)(2n-1)}{n-2}}$
$\mathcal{D}$	$[2(1-n), 2(n-1)^2, 0]$	$\begin{cases} a = \frac{kt}{2n^2-2n-1} & \text{if} \quad k \neq 0 \\ a = a_0 t & \text{if} \quad k = 0 \end{cases}$
$\mathcal{E}$	$[-1-3\omega, 0, -1-3\omega]$	$a = a_0(t - t_0)$
$\mathcal{F}$	$[1-3\omega, 0, 2-3\omega]$	$a = a_0(t - t_0)^{1/2}$ (only for $n = 3/2$)
$\mathcal{G}$	$\left[-\frac{3(n-1)(1+\omega)}{n}, \frac{(n-1)[4n-3(\omega+1)]}{2n^2}, \frac{n(13+9\omega)-2n^2(4+3\omega)-3(1+\omega)}{2n^2}\right]$	$a = a_0\, t^{\frac{2n}{3(1+\omega)}}$



The dynamical systems approach

$$x = \frac{\dot{f}'}{f'H}, \qquad y = \frac{R}{6H^2}, \qquad z = \frac{f}{6f'H^2},$$

$$\Omega = \frac{\mu_m}{3f'H^2}, \qquad K = \frac{k}{a^2H^2}, \quad \mathfrak{q} \equiv \left(\frac{d\log F}{d\log R}\right)^{-1} = \frac{f'}{Rf''}.$$

Autonomous set of equations

$$\begin{aligned}
\frac{dx}{dN} &= \varepsilon\left(2K + 2z - x^2 + (K + y + 1)x\right) + \Omega\varepsilon\left(-3w - 1\right) + 2,\\
\frac{dy}{dN} &= y\varepsilon\left(2y + 2K + x\mathfrak{q} + 4\right),\\
\frac{dz}{dN} &= z\varepsilon\left(2K - x + 2y + 4\right) + x\varepsilon\, y\mathfrak{q},\\
\frac{d\Omega}{dN} &= \Omega\varepsilon\left(2K - x + 2y - 3w + 1\right),\\
\frac{dK}{dN} &= K\varepsilon\left(2K + 2y + 2\right),
\end{aligned}$$

Constraint $\Rightarrow$ $1 = -K - x - y + z + \Omega\,,$

- Carloni, Dunsby Capozziello, Troisi (CQG, 2005)
- Amendola et. al. (PRD, 2007)
- Carloni, Dunsby, Troisi (GRG, 2009)

Large-scale density perturbations

After some calculation we find that large-scale density perturbations evolve according to the following 3rd order equation:

$$(n-1)\dddot{\Delta}_m - (n-1)\left(\frac{4n\omega}{\omega+1} - 5\right)\frac{\ddot{\Delta}_m}{t} + \mathcal{D}_1(n,w)\,\frac{\dot{\Delta}_m}{t^2} + \mathcal{D}_2(n,w)\frac{\Delta_m}{t^3}$$

$$+ \mathcal{D}_3(n,w)\,\mathcal{C}_0\,t^{-\frac{4n}{3(\omega+1)}-1} = 0$$

$$\Delta_m = K_1 t^{-1} + K_2 t^{\alpha_+|_{w=0}} + K_3 t^{\alpha_-|_{w=0}} - K_4\frac{\mathcal{C}_0}{S_0^2}t^{2-\frac{4n}{3}},$$

$$\alpha_\pm|_{w=0} = -\frac{1}{2} \pm \frac{\sqrt{(n-1)(n(32n(8n-19)+417)-81)}}{6(n-1)}\,,$$

$$K_4|_{w=0} = \frac{9(n(12n-31)+18)}{8(4n-9)\,(12n^3-19n^2-3n+9)}\,.$$

Nontrivial dependence on n

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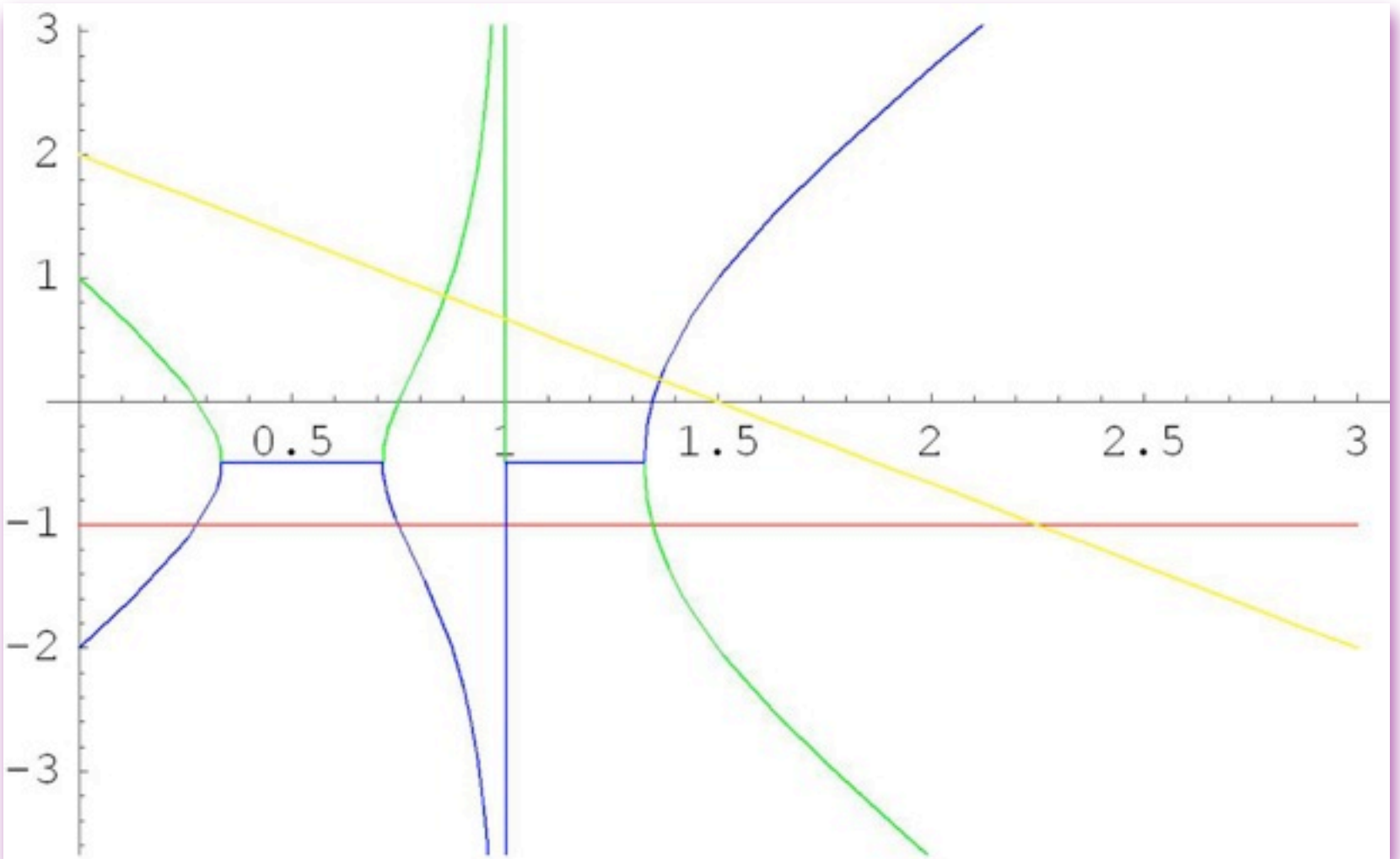
Nontrivial dependence on n

The long wavelength perturbations grow for all values of n, even for an accelerated background!



The Matter Power Spectrum

An important quantity to characterized the small scale perturbations in the power spectrum

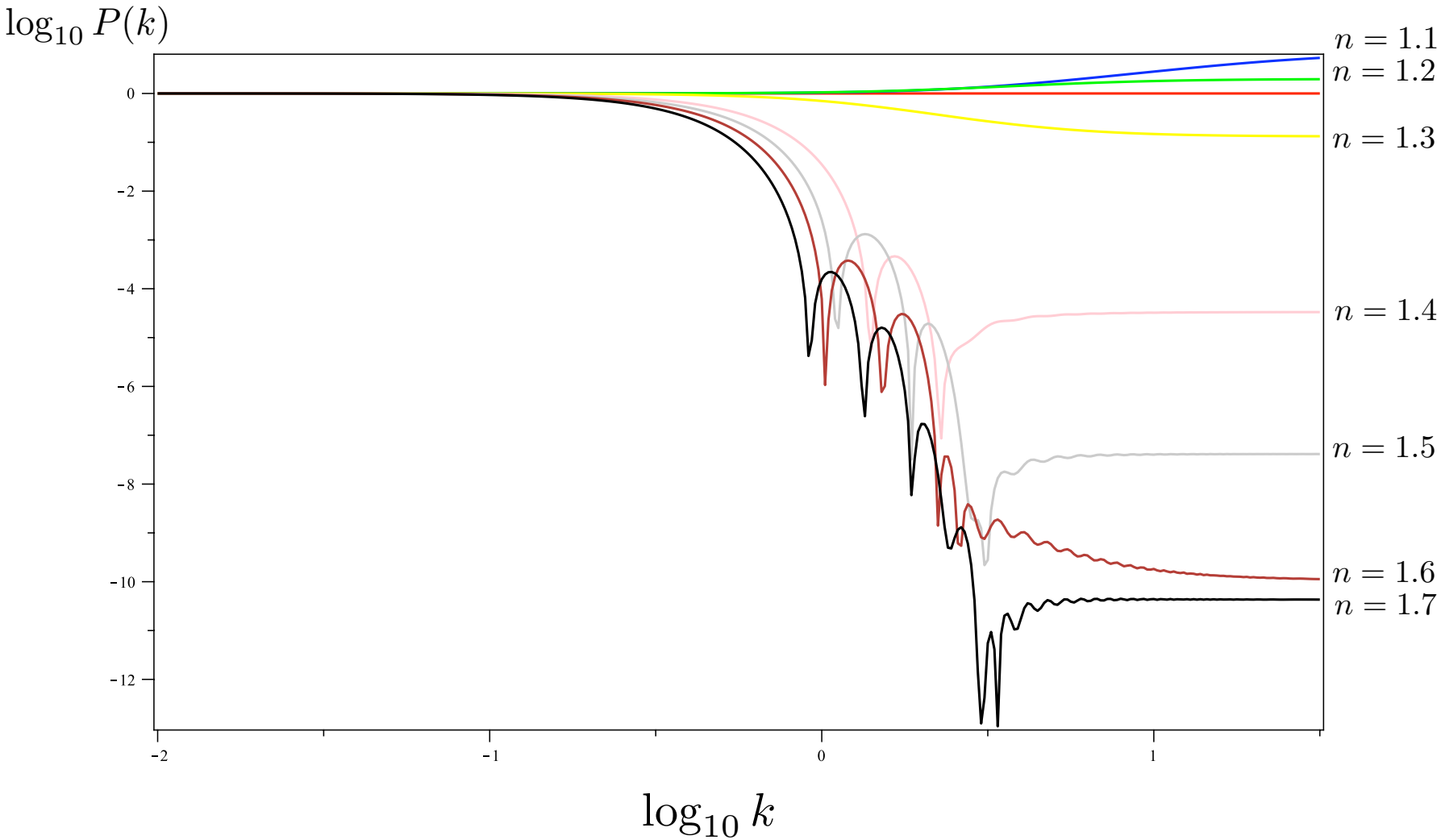
$$\langle \Delta_m(\mathbf{k}_1)\Delta_m(\mathbf{k}_2)\rangle = P(k_1)\delta(\mathbf{k}_1+\mathbf{k}_2)$$

This quantity tells us how the fluctuations of matter depend on the wave number at a specific time and carries information on the amplitude of the perturbations on a given scale.

In GR the power spectrum on large scales is constant, while on small scales it is suppressed depending on the nature of the cosmological fluid(s).

The case of pure dust is special: perturbations are scale invariant.

The Power Spectrum for R^n-gravity (n>1)



On large and small scales the spectrum is scale-invariant.

Oscillations can occur around a specific value of k depending on the parameter "n".

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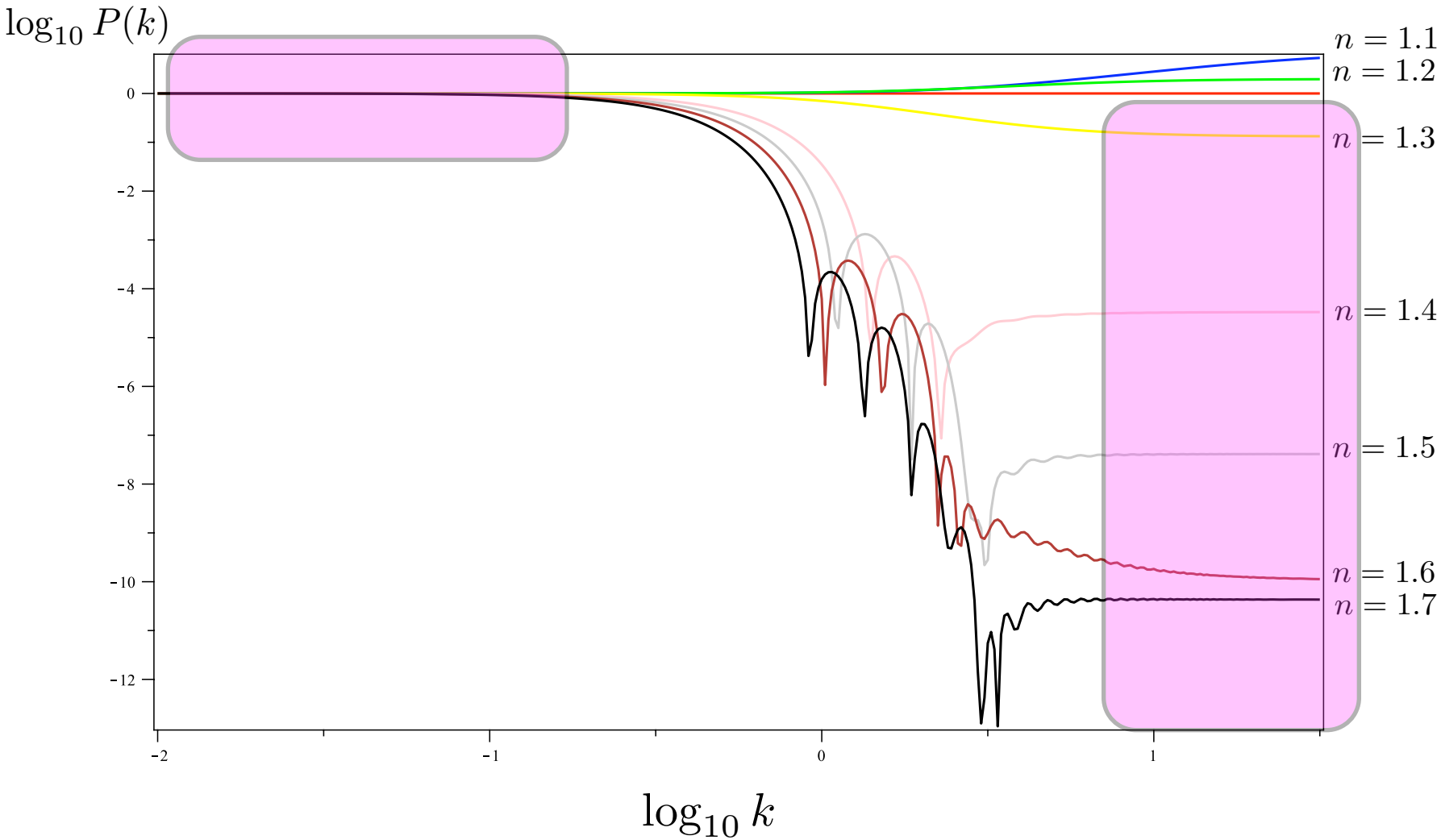

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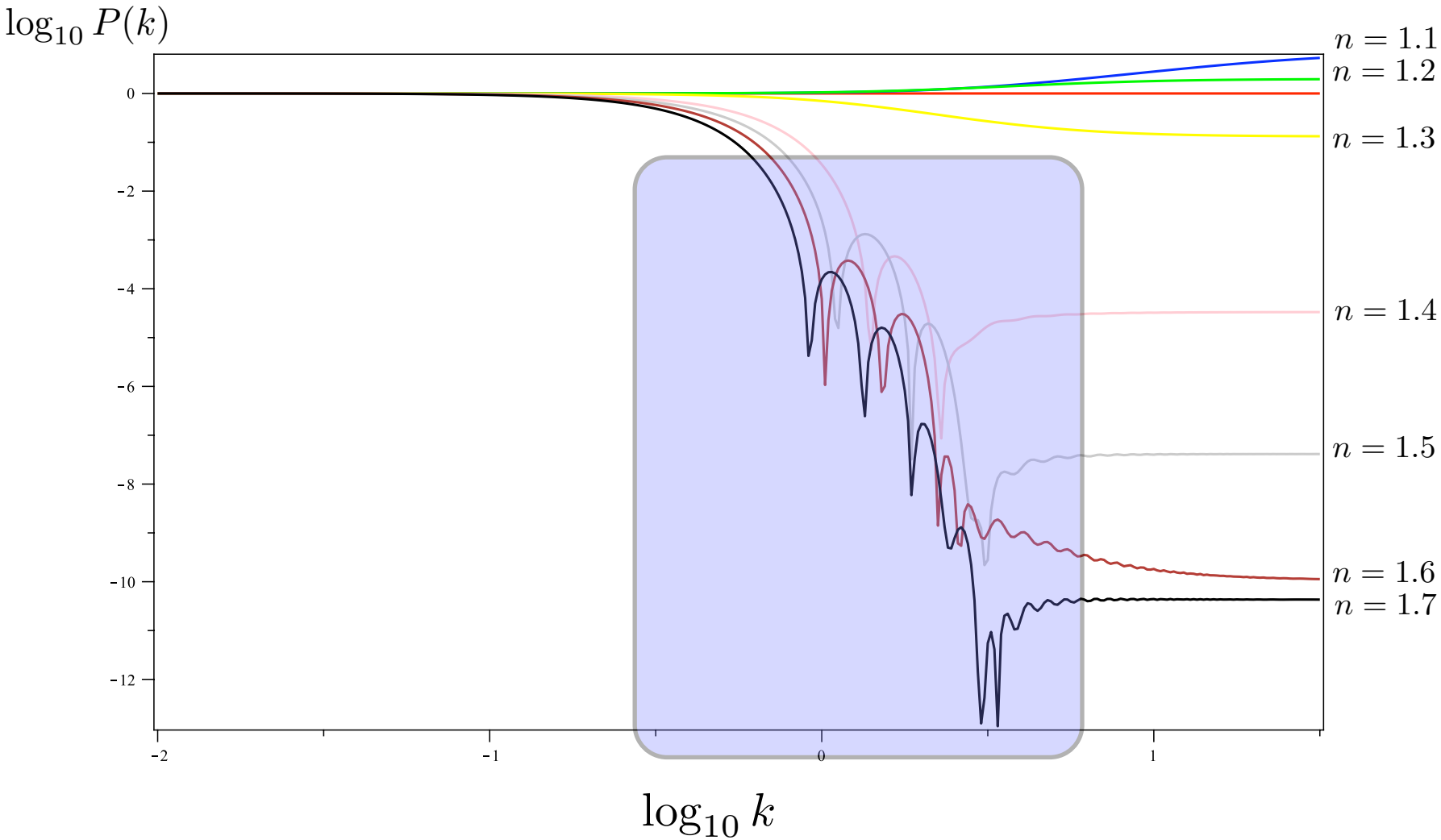

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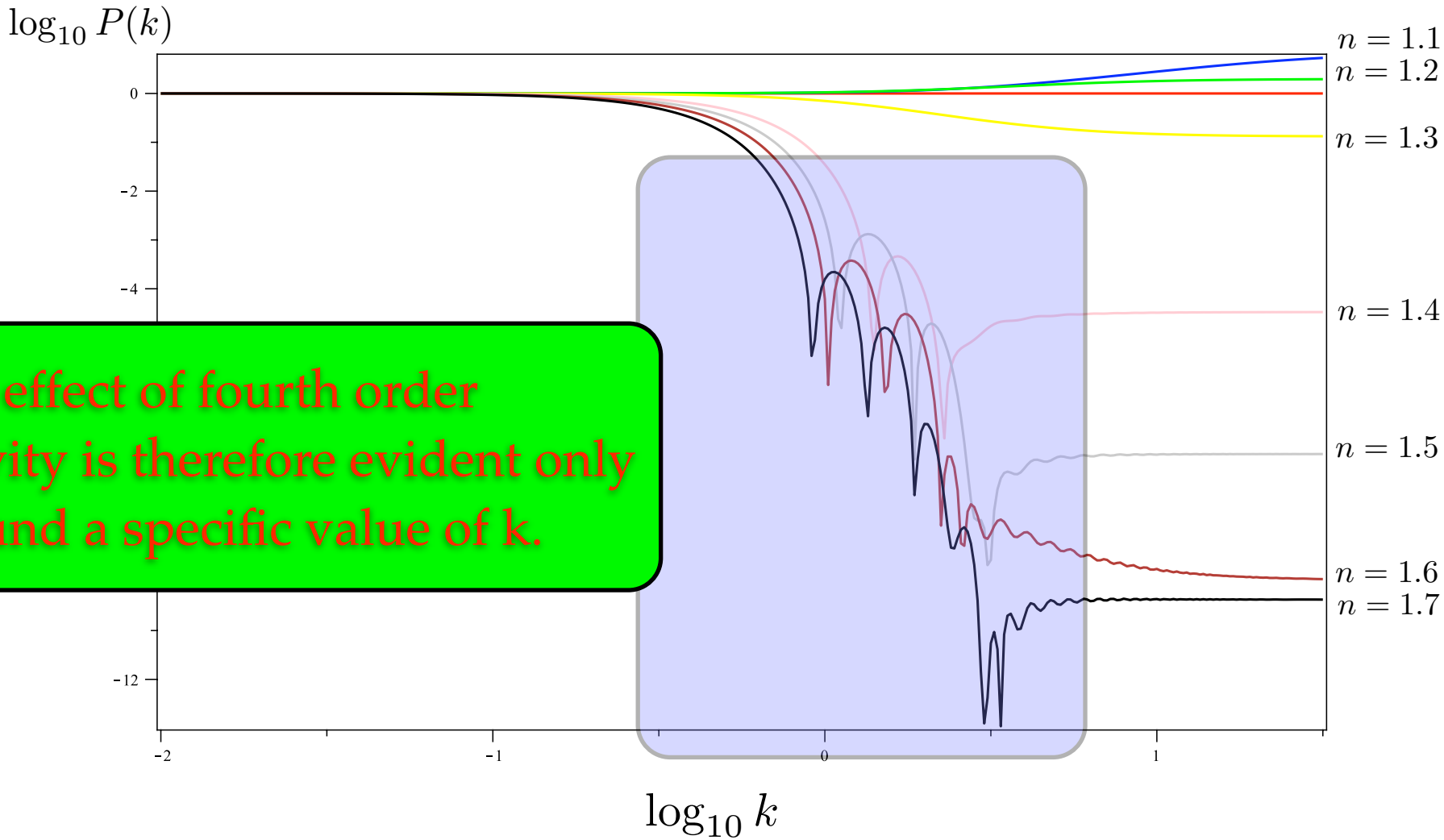

The effect of fourth order gravity is therefore evident only around a specific value of k.

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Oscillations can occur around a specific value of k depending on the parameter "n".

Is this result general?

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- The k-structure of the perturbation equations is independent of the theory of gravity,
- The interaction between fourth order gravity and matter is peaked at certain specific scales and becomes k-independent on large and small scales.
- WORK IN PROGRESS with other models.
- PROBLEM: we don't really know much about their background.

IF verified, this result would constitute a clear and relatively easy way to probe fourth order gravity on cosmological scales.

Extending 1+3 to 1+1+2

- 1+3 spacetime split adopted to perturbations of cosmological backgrounds.
- Many astrophysical systems have high degree of symmetry.
- This suggests further decomposition of the 3 spatial degrees of freedom relative to preferred spatial vector.
- We end up with a "1+1+2" split of spacetime.
- This provides an excellent framework for studying spherically symmetric (SS) spacetimes and their perturbations.

The basic framework

Unit vector n^a, $n^a n_a = 1$, $u^a n_a = 0$,

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Decomposition

$$\psi^a = \Psi n^a + \Psi^a$$

$$\psi_{ab} = \psi_{\langle ab\rangle} = \Psi\left(n_a n_b - \tfrac{1}{2}N_{ab}\right) + 2\Psi_{(a}n_{b)} + \Psi_{ab}$$

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Derivatives

$$\hat{\psi}_a \equiv n^e \tilde{\nabla}_e \psi_a,$$

$$\delta_e \psi_a \equiv N_a{}^f N_e{}^j \tilde{\nabla}_j \psi_f$$

Decomposition of basic quantities

Kinematics of n^a :

$$\tilde{\nabla}_a n_b = n_a a_b + \tfrac{1}{2}\phi N_{ab} + \xi\varepsilon_{ab} + \zeta_{ab},$$

Acceleration, expansion, twist and shear of "sheet"

$$\begin{aligned} a_a &\equiv n^c\tilde{\nabla}_c n_a = \hat{n}_a, \\ \phi &\equiv \delta_a n^a, \\ \xi &\equiv \tfrac{1}{2}\varepsilon^{ab}\delta_a n_b, \\ \zeta_{ab} &\equiv \delta_{\{a} n_{b\}}. \end{aligned}$$

Kinematics

$$\begin{aligned} \dot{u}^a &= \mathcal{A}n^a + \mathcal{A}^a, \\ \omega^a &= \Omega n^a + \Omega^a, \\ \sigma_{ab} &= \Sigma\left(n_a n_b - \tfrac{1}{2}N_{ab}\right) + 2\Sigma_{(a}n_{b)} + \Sigma_{ab}, \end{aligned}$$

Weyl

$$\begin{aligned} E_{ab} &= \mathcal{E}\left(n_a n_b - \tfrac{1}{2}N_{ab}\right) + 2\mathcal{E}_{(a}n_{b)} + \mathcal{E}_{ab}, \\ H_{ab} &= \mathcal{H}\left(n_a n_b - \tfrac{1}{2}N_{ab}\right) + 2\mathcal{H}_{(a}n_{b)} + \mathcal{H}_{ab}. \end{aligned}$$

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LRS condition

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Equations for SS spacetimes

Field Equations

$$
\begin{aligned}
\hat{\phi} &= -\frac{1}{2}\phi^2 + \frac{1}{3}R - \frac{2}{3}\frac{f}{f'} + \frac{f''}{f'}X\left(\phi + 2\mathcal{A}\right) + A\phi \,, \\
\hat{\mathcal{A}} &= -\mathcal{A}^2 - \mathcal{A}\phi + \frac{1}{6}\frac{f}{f'} - \frac{1}{3}R - \frac{f''}{f'}X\mathcal{A} \,, \\
\hat{R} &= X \,, \\
\hat{X} &= -\frac{1}{3}\frac{Rf'}{f''} + \frac{2}{3}\frac{f}{f''} - X\phi - \frac{f'''X^2}{f''} - X\mathcal{A} \,.
\end{aligned}
$$

Null geodesics

$$
\begin{aligned}
E' &= -E^2\mathcal{A}\kappa \,, \\
\kappa' &= E(1-\kappa^2)\left(\frac{1}{2}\phi - \mathcal{A}\right)
\end{aligned}
$$

Some general results

Condition for the existence of solutions with constant scalar curvature is:

$$2f(R_0) - R_0 f'(R_0) = 0\,.$$

Barrow and Ottewill (1983)

Condition for the existence of the Schwarzshild solution is:

$$f(0)/f'(0) = 0.$$

This holds for many (but not all) classes of theories.... e.g,

$$R^n,\ R + \alpha R^n,\ R/(1 + AR)$$

The 1+1+2 equations can be used to generate exact SS solutions → test violation of Birkhoff.

Goswami et al, 2009... on axXiv very soon.

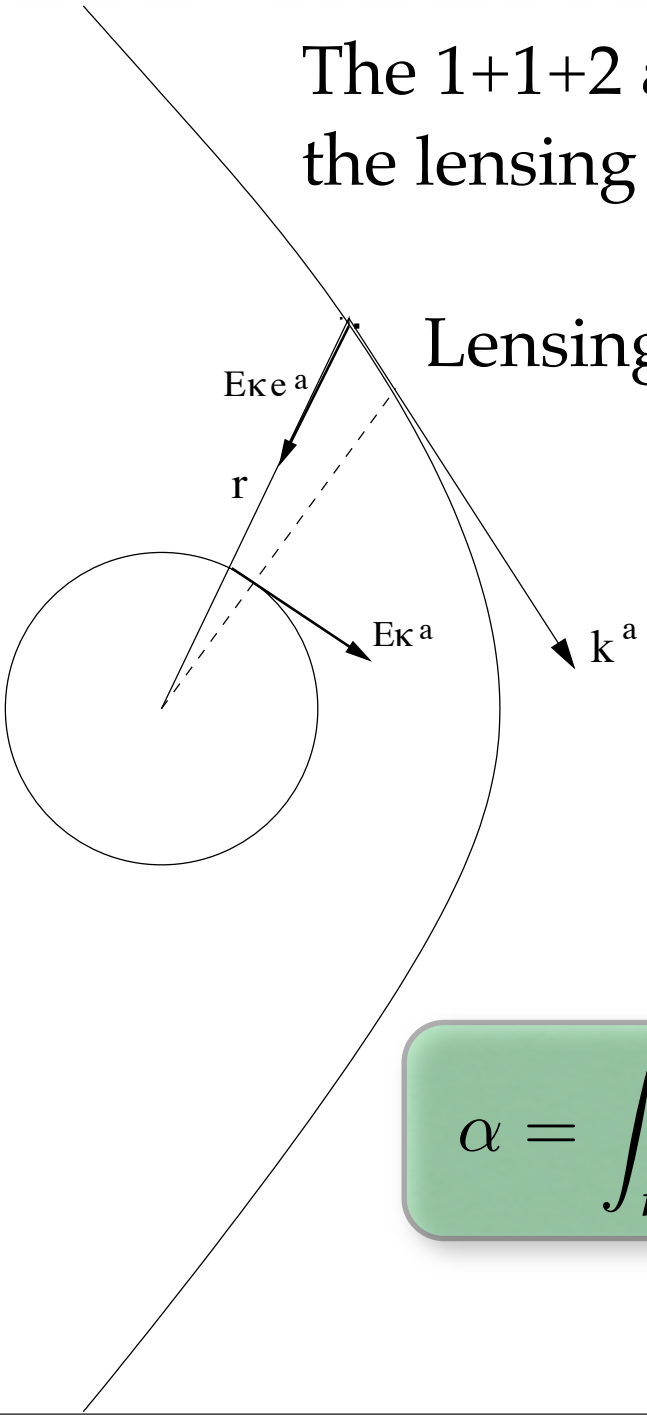
Gravitational Lensing in f(R) gravity

The 1+1+2 approach is ideal for studying the lensing problem

Lensing geometry

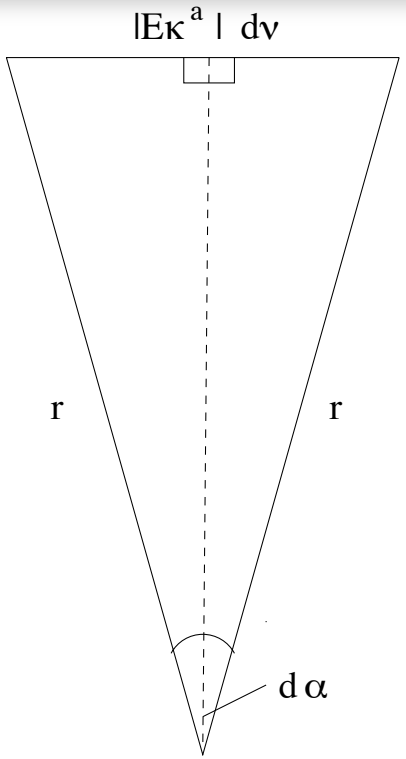
$$n^a = \kappa e^a + \kappa^a$$

$$k^a = E(u^a + \kappa e^a + \kappa^a)$$



$$d\alpha = \frac{1}{r}\,|E\kappa^a|\,d\nu$$

$$\alpha = \int_{\nu_1}^{\nu_2} \frac{1}{r}\,|E|\,\sqrt{\kappa^a \kappa_a}\,d\nu - \alpha_0$$



An example: Clifton's solution

A number of exact solutions exist, e.g., Clifton (2006)

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

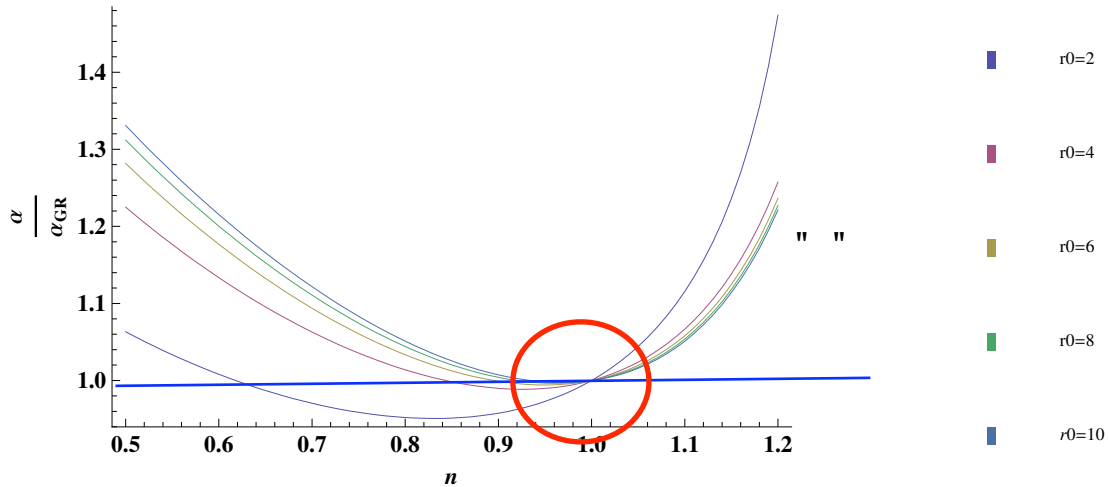
$$A(r) = r^{(2n-2)\frac{(2n-1)}{(2-n)}} + \frac{C}{r^{\frac{(5-4n)}{(2-n)}}}$$

$$\frac{1}{B(r)} = \frac{(2-n)^2}{(7-10n+4n^2)(1+2n-2n^2)}\left(1 + \frac{C}{r^{\frac{(7-10n+4n^2)}{(2-n)}}}\right)$$

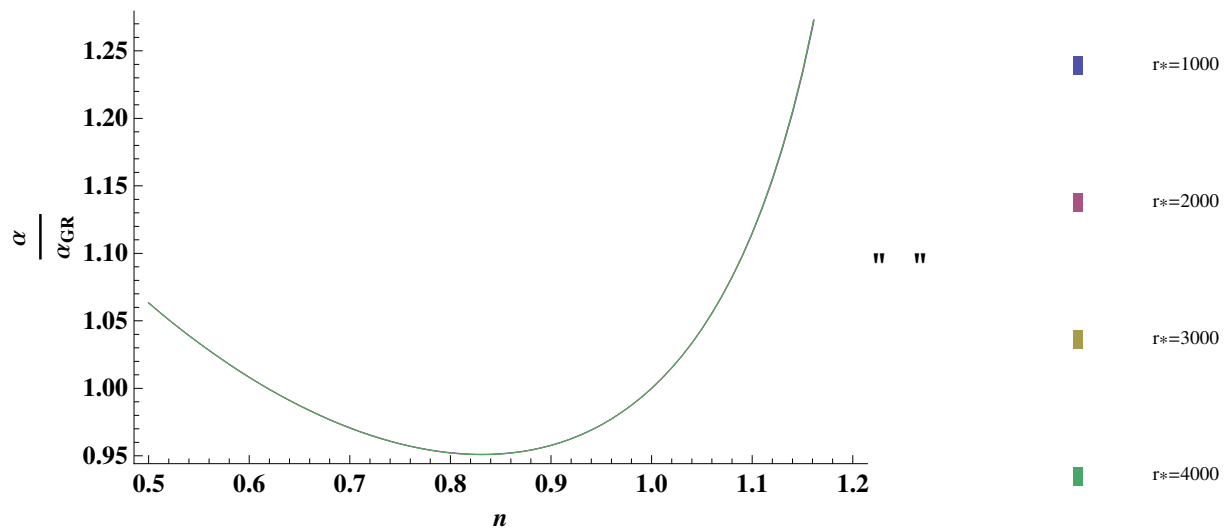
The bending angle can be easily found:

$$\alpha = 2\int_{r_0}^{r_*} L^{-1}\frac{J}{r^2}[r^{\frac{(4n^2-6n+2)}{n-2}} - J^2(r^{-2} + Cr^{\frac{(4n^2-12n+11)}{n-2}})]^{-\frac{1}{2}}dr - \pi$$

de Swart et al, Nkioki et al, 2009 (see arXiv soon)



For values of n close to 1, the bending angle only differs from the GR result by a few percent.



This shows that the bending angle is independent of r_*

The Future.....

- Extend to multi-fluid systems e.g. CDM, Baryons + Radiation (Abebe & Abdelwahab),
- Generate spherically symmetric solutions (Goswami et al),
- The evolution of null geodesics in f(R) gravity - observational relations etc.
- The Newtonian limit and Newtonian perturbation theory
- and much much more if f(R) survives observational scrutiny!!

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